On the Essentiality of Electronic Money*

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Preliminary and Incomplete, Do not Circulate

Abstract

What makes *e-money* special relative to cash? How does the introduction of electronic money improve the functioning of a payment system, in which cash is used as an incumbent payment instrument? This paper adopts a mechanism design approach to identify the essential features of e-money that improve the efficiency of the economy. We build a micro-founded general equilibrium model to compare the efficiency properties of different payment systems. We start with a basic environment in which traditional cash is used as a payment instrument. We then gradually attach additional features to this payment instrument, including some distinctive characteristics of electronic money. We identify several features of e-money which can help mitigate fundamental frictions and enhance social welfare in a cash economy. Our findings help clarify how e-money can be an essential component of a payment system (even in the absence of other transactions cost advantage over cash), and provide guidance on the design and regulation of e-money products.

Keywords: money and electronic money, retail payment, mechanism design, search and matching.

*The views expressed here do not necessarily reflect the position of the Bank of Canada.

1 Introduction

Recent years have witnessed a number of retail payment innovations[1] known as electronic money (or *e-money*).[2] The latest generation of electronic money substantially improves the performance of payment instruments in terms of convenience, durability, and transaction speed. Over the course of history, emergence and adoption of new medium of exchange - for example from Yap stones to seashells to paper money - have been taking place, but the basic functioning of the payment system for monetary exchange remains largely unchanged. In a monetary payment system, no matter Yap stones or paper money is used, in order to purchase a product, a buyer needs to first acquire a means of payment from other agents, bring it to the point of sale, and then conduct a quid-pro-quo exchange with the seller, who then uses it in other transactions. These observations seem to suggest the existence of deep, fundamental frictions that underlie and determine the basic mode of monetary exchange. Payment technologies have certainly evolved over time, it is unclear though whether all of these improvements are useful for overcoming the deep frictions that shape the basic functioning of payment systems. The emergence of e-money provides an opportunity for understanding and answering some basic but important questions about payment system: is e-money merely another kind of seashells, or instead something fundamentally different from money, something helps mitigate deep frictions? What are these frictions? How do various payment systems emerge endogenously in response to these frictions?

To answer these questions, this paper is built on recent developments in monetary theory. It is now widely recognized that when there are deep frictions like the lack of commitment and lack of recordkeeping, the use of money as payment instrument improves the efficiency of resource allocations (Kocherlakota, 1998). In this sense, money, as a medium of exchange, is *essential* because it improves efficiency relative to an economy without money. However, modern monetary theory also teaches us that, in a world subject to frictions that render money essential, equilibrium allocation is typically suboptimal. This is because the use of money requires pre-investment by impatient buyers, giving rise to a cash-in-advance constraint. In a decentralized economy, this constraint leads to an inefficient allocation: impatient buyers acquiring too little money, and hence are liquidity constrained in trading (for example due to discounting and inflation).

[1] The Survey of Electronic Money Developments by the CPSS noted that "in a sizeable number of the countries surveyed, card-based e-money schemes have been launched and are operating relatively successfully: Austria, Belgium, Brazil, Denmark, Finland, Germany, Hong Kong, India, Italy, Lithuania, the Netherlands, Nigeria, Portugal, Singapore, Spain, Sweden and Switzerland. In some countries the products are available on a nationwide basis and in others only within specific regions or cities ... Compared to card-based schemes, the developments of network-based or software-based e-money schemes has been much less rapid. Network-based schemes are operational or are under trial in a few countries (Australia, Austria, Colombia, Italy, the United Kingdom and the United States), but remain limited in their usage, scope and application." (CPSS, 2001)

[2] There is no universal definition for e-money that can fit precisely all exisiting variants of e-money products. One definition of e-money proposed by the Committee on Payment and Settlement Systems (CPSS) is the following: it is the "monetary value represented by a claim on the issuers which is stored on an electronic device such as a chip card or a hard drive in personal computers or servers or other devices such as mobile phones and issued upon receipt of funds in an amount not less in value than the monetary value received and accepted as a means of payment by undertakings other than the issuer." This defintion is quite broad (e.g. including debit cards), and at the same time quite narrow (e.g. excluding Bitcoin). For the purpose of this paper, we don't need to stick with one specific definition of e-money. Instead, we will examine below several features that are commonly found in e-money products.

In addition, resource misallocation can be magnified by an inefficient trading mechanism, since the trade surplus is not divided in a way respecting the pre-investment of buyers. All these effects give rise to the so-called "holdup" problem (Lagos and Wright, 2005).

The aforementioned frictions that render money essential also shape the basic functioning of monetary payment systems. Owing to its full anonymity and decentralization of trades, the conventional money-based payment system typically exhibits the following features. First, *non-exclusive participation.* Anyone can freely participate in the payment system to transfer or receive money balances. Without other prerequisite, anyone can receive a transfer of balances from others, and anyone can transfer to others part or all of the balances one possesses. Second, *unrestricted transferability.* Beyond transaction costs, there is no restriction on the transferability of money balances. Any amount of money can be transferred anytime, anywhere between any parties. In other words, non-exclusive participation means no limitation on who can use money (the extensive margin), and unrestricted transferability means no limitation on how money is used (the intensive margin). In addition, all transfers are *zero-sum:* the amount of money balances transferred by the payer is always equal to the amount received by the payee.

We argue that an e-money-based payment system is fundamentally different from money because it can be free of these features. First, e-money issuers can exclude participation. E-money system typically requires payers and payees to obtain specific payment devices from the e-money issuer before they can hold and transfer balances (e.g. cards, electronic devices, software for consumers, and card reader/writer for merchants). Noncompliance leads to exclusion from the system. Second, e-money technology often requires e-money balances to be associated with an account (e.g. prepaid card) and hence allows the e-money issuer to restrict balance transfers. In addition, since balances are transferred through electronic devices, it is technically feasible to have non-zero-sum transfers: the amount of balances transferred by the payer differs from the amount received by the payee. This feature of e-money may allow for charging merchants fees or other transaction fees, which are often observed in e-money payment systems. [3]

Of course, the fact that e-money is fundamentally different from money does not necessarily mean it is more essential. Next we use a mechanism design approach to identify the distinctive features of e-money that are also essential. We build a micro-founded general equilibrium model to compare the efficiency properties of different payment systems. The starting point is a basic environment in which traditional cash is used as a payment instrument. We then gradually attach additional features to this payment instrument, including some distinctive characteristics of e-money. We identify several features of e-money which can help mitigate fundamental frictions and enhance efficiency in a cash economy. First, we consider e-money featuring *limited participation.* The technical possibility of excluding non-compliant

[3] Other payments systems (such as banking, credit cards and large-value settlement systems) may also exhibit these features. But these systems usually require some centralized arrangements monitored by banks, credit card issuers or clearing houses who possess a richer information set and/or a stronger enforcement technology than an e-money issuer. It is not clear (i) whether these centralized arrangement are feasible in the current environment; and (ii) whether money and the e-money remain essential in an environment in which these centralized arrangements (e.g. credit) are feasible. In this regard, the model constructed in this paper may not be a good one for analyzing these systems.

traders allows e-money issuers to enforce pre-trade transfers (e.g. membership fees for obtaining e-money devices). Second, we consider an e-money featuring *limited transferability.* The technical possibility of limiting transferability and having non-zero-sum transfers allow the e-money issuer to enforce post-trade charges (e.g. merchant fees and interchange fees).

We show that the introduction of e-money may help relax certain binding constraints faced by the money-issuer and allow more flexible and efficient intervention. The first main finding of our paper is that inefficient allocation can arise even in an *optimally designed* monetary system (subject to non-exclusive participation, unrestricted transferability, and zero-sum transfers). The second main finding of our paper is that these new features of e-money are essential because they help achieve cross-subsidization between buyers and sellers and improve efficiency in resource allocation relative to a payment system without these features. In addition, we show that e-money with limited transferability is more powerful than one with limited participation. Finally, we characterize some key properties of optimal e-money mechanisms, and provide some examples of simple direct and indirect mechanisms that can achieve first-best.

While developments in payment raise new policy issues for central banks and regulators, so far there has been limited guidance provided by economic theory regarding the welfare implication of e-money adoption.[4] To the best of our knowledge, no existing research on e-money performs welfare analysis with serious consideration of fundamental frictions in payment systems. For one thing, modern monetary theory focuses on understanding the fundamental roles of conventional money and credit, while the role of potential features brought by e-money has not yet been fully explored. Our paper is also the first one that develops a micro-founded general equilibrium model of e-money. By uncovering essential features of a payment instrument like e-money, our results can provide guidance to payment system designers and regulators on whether and how these products should be regulated, minimizing the distortion on critical features of these products.

Our paper is directly related to several lines of research in the monetary literature. This literature underscores the fact that money is intrinsically worthless means of payments. Hence the demand for these objects are derived from the future consumption values of those goods and services for which they can exchange. Therefore, the liquidity values of money and e-money should be determined endogenously in a dynamic, general equilibrium setting. Lagos and Wright (2005) and Rocheteau and Wright (2005) develop a useful framework for studying money and monetary policy in a very tractable fashion. Recent models of payments system building on this framework include Monnet and Roberds (2008) and Li (2011). As mentioned above, payment systems in these economies usually cannot implement the socially optimal allocation (the first best), due to the holdup problem. We highlight two strands of researches on the implementation of the first best within a payment system. One strand takes an inefficient trading protocol as a primitive, and studies the design of monetary policy to mitigate this inefficiency. For example, Lagos

[4]For example, in the Survey of Electronic Money Development, the Bank for International Settlement highlighted that "Electronic money projected to take over from physical cash for most if not all small-value payments continues to evoke considerable interest both among the public and the various authorities concerned, including central banks." (CPSS, 2001)

and Wright (2005) and Lagos (2010) find that the Friedman rule is optimal in these environments, but it involves taxing agents, which is not incentive compatible as agents will not voluntarily pay taxes. With the use of a fixed fee and linear transfers, Andolfatto (2010) illustrates how the first best can be implemented with voluntarily participation in a competitive environment. There is also another strand of researches, for example Hu, Kennan, and Wallace (2009) and Rocheteau (2012), which takes the inefficiency in monetary policy as given, and designs the trading protocol to mitigate this inefficiency. These studies endogenize the trading protocol with a mechanism design approach, as advocated by Wallace (2010). This literature finds that, under certain conditions, the first best can still be implemented by adopting an optimal trading protocol in pairwise trades. Specifically, deviation from Friedman's rule can still be optimal, and the welfare cost of inflation can be zero. Our paper is related to both strands of researches. Unlike Lagos and Wright (2005) or Andolfatto (2010), we do not restrict to any particular type of intervention, and use mechanism design approach to endogenize the payment instruments and payment system. Unlike Hu, Kennan, and Wallace (2009) or Rocheteau (2012), we take inefficient trading protocols as one of the primitive input to the mechanism design of payment system. Our perspective is particularly relevant for policy makers, such as central banks and payment system regulators, who arguably have limited influence over the determination of terms-of-trade in a decentralized and anonymous situation. The mechanism design approach is powerful since it can help to identify the essential features of e-money and clarify their role in the payment system.

Our paper is also related to the two-sided market literature that emphasizes the importance of network externalities in the adoption of payment instruments. This component plays a crucial role in our model and generates interesting implications in terms of efficiency and the optimal design of payment system. See Rochet and Tirole (2003), Armstrong (2006) and Weyl (2010) for the prototype model of platforms and their competition. There is also a large strand of researches which study electronic payment platform like credit cards, for example Gowrisankaran and Stavins (2004), Humphrey, Pulley and Vesala (1996), Shy and Tarkka (2002), Shy and Wang (2010) and Wright (2003). Gans and King (2003) study a platform model of credit card with the presence of cash users. Most of these researches focus on the positive theories of the fee structure and the competition between profit-maximizing platforms. Our paper complements these researches by studying the normative aspect of payment instruments when they can be used by a welfare-maximizing mechanism designer as policy vehicles.

The rest of the paper is organized as follows. Section 2 presents the model environment. Section 3 designs the optimal money mechanism, highlighting the importance of non-linear schemes and its limitation. Section 4 designs the optimal e-money mechanism with limited participation, highlighting the importance of cross-subsidization and its limitation. Section 5 designs the optimal e-money mechanism with limited transferability, highlighting the importance of after-trade fees and its essentiality. Section 6 concludes.

2 Baseline Model

Our model bases on the alternating market formulation from Lagos and Wright (2005). Time is indexed by $t = 0, 1....$ Each period is divided into two consecutive subperiods: day and night. There are two non-storable goods in this economy. Consumption goods produced and consumed during the day, and numeraire goods at night. The economy is populated with two types of infinitely lived agents: measure one of buyers and measure one of sellers. Agents discounts the future and have a discount factor $\beta \in (0, 1)$. Buyers and sellers are subject to pairwise random matching in the day. This decentralized market is denoted by DM. The market at night is centralized and competitive, called CM. In the day, buyers value DM goods q with utility $U(q)$, which satisfies $U' > 0, U'' < 0, U(0) = 0$ and $\lim_{q \to 0} U'(q) = \infty$. In the night, buyers have access to a linear production technology according to which l units of work generate l units of numeraire, and they are endowed with sufficiently large $\bar{l} < \infty$ units of labor per period such that the labor supply is never bounded.[5] Buyers have the following preferences over consumption goods q_t and labor l_t:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{U(q_t) - l_t\}.$$

In the day, sellers have access to a production technology $C(q)$ according to which $C(q)$ units of work generate q units of consumption goods. The cost function satisfies $C(0) = 0, C'(q) \geq 0, C''(q) \geq 0, C'(0) = 0$. In the night, sellers value numeraire according to a linear function, and they are also endowed with sufficiently large $\bar{l} < \infty$ units of labor per period such that the output is never bounded. A seller has the following preferences over producing consumption goods q_t and consuming numeraire l_t:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t \{-C(q_t) + l_t\}.$$

Meetings in the day are anonymous, and hence there is no scope for credit in this market. Transactions in the DM must be quid-pro-quo. In this economy, there is an additional, perfectly divisible storable objects called *money* (or cash) that can be used as medium of exchange. There is a money-issuer (e.g. central bank) who controls the supply of money M_t. The gross growth rate of the money stock across two periods is $\mu_t = M_t/M_{t-1}$. Money and the numeraire goods are traded in the CM. Agents take the price of money in terms of the numeraire good, ϕ_t, as given. So the aggregate real balances of money in the economy is $\phi_t M_t$. In the baseline setting, money growth is implemented by lump sum transfers T_t to buyers such that[6]

$$T_t = (\mu - 1)\, \phi_t M_t. \tag{1}$$

[5] In Lagos and Wright, the CM preference is quasi-linear instead of linear. This difference does not matter for our result.

[6] Here, we assume that the lump sum money transfer is given only to buyers. Our result will not change if both buyers and sellers receive the transfer.

We focus on symmetric and stationary equilibria with constant money growth μ where agents of the same type follow identical strategies and where real allocations are constant over time. In stationary equilibrium, end-of-period real money balances are time-invariant, $\phi_t M_t = \phi_t M_t$, implying $\phi_t/\phi_{t+1} = M_t/M_{t+1} = \mu$

Denote the nominal money balances that buyers and sellers bring to the period t DM by $m_{b,t}$ and $m_{s,t}$ respectively. The real balances are $\tilde{z}_b \equiv \phi_t m_{b,t}$ and $\tilde{z}_s \equiv \phi_t m_{s,t}$. When a buyer with $\tilde{z}_b$ matches with a seller with $\tilde{z}_s$ in the DM, he pays $d(\tilde{z}_b, \tilde{z}_s)$ balances to buy $q(\tilde{z}_b, \tilde{z}_s)$ DM goods, with the terms of trade determined by proportional bargaining, which is described in detail shortly. We conjecture now and verify later that the terms of trade depend only on the buyer's balance. We can thus omit the $\tilde{z}_s$ argument and rewrite the terms of trade as $q(\tilde{z}_b)$ and $d(\tilde{z}_b)$ .While $\tilde{z}_b, \tilde{z}_s$ denote the real balances of agents at the beginning of a DM, the real balances at the end of the CM is just $z_b \equiv \mu\tilde{z}_b, z_s \equiv \mu\tilde{z}_s$.[7] Below, we will use $\tilde{z}_j$ to denote beginning of DM money balances and z_j to denote end of CM money balances. The allocation in stationary equilibrium is given by (q, d, z_b, z_s).

Value Functions

We denote the value functions of a type $j = b, s$ with z in the CM and with $\tilde{z}$ in the DM by $W_j\ (z)$ and $V_j\ (\tilde{z})$. Consider first a buyer with $\tilde{z}$ in the DM. Her value function $V_b(\tilde{z})$ is

$$V_b(\tilde{z}) = U[q(\tilde{z})] + W_b[z - d(\tilde{z})]. \tag{2}$$

In the CM, the buyer's value function, $W_b(z)$, is

$$W_b(z) = \max_{z',l} \left\{ -l + \beta V_b\left(\frac{z'}{\mu}\right) \right\},$$

subject to

$$z + l + T \geq z'.$$

Here, a buyer chooses money balances z' to be brought into the DM which is financed by initial money holding z, CM labor l and transfer from the central bank. This problem can be simplified to

$$W_b(z) = z + T + \max_{z'} \left\{ -z' + \beta V_b\left(\frac{z'}{\mu}\right) \right\}, \tag{3}$$

Similarly, the seller's DM value function is

$$V_s(\tilde{z}) = -C[q(\tilde{z}_b)] + W_s[\tilde{z} + d(\tilde{z}_b)]. \tag{4}$$

[7] This is because $z_b = \phi_t m_{b,t+1} = \mu\phi_{t+1} m_{b,t+1} = \mu\tilde{z}_b$. Similarly $z_s = \mu\tilde{z}_s$.

In the CM, the seller's value function is

$$W_s(z) = z + \max_{z'} \left\{ -z' + \beta V_s\left(\frac{z'}{\mu}\right) \right\}, \tag{5}$$

where the CM consumption is $l = z - z'$. Note that both $W_b(z)$ and $W_s(z)$ are linear in z. This immediately implies that $V_s(\tilde{z})$ is also linear.

Decision in DM

The terms of trade $\{q(\tilde{z}), d(\tilde{z})\}$ in the DM is determined by proportional bargaining, whereby a buyer gets a fraction θ of the total trade surplus in the match. Consider a match in the DM. Buyer's trade surplus is

$$S_b(q, d; \tilde{z}_b, \tilde{z}_s) \equiv U(q) + W_b(\tilde{z}_b - d) - W_b(\tilde{z}_b),$$

and seller's trade surplus is

$$S_s(q, d; \tilde{z}_b, \tilde{z}_s) \equiv -C(q) + W_s(\tilde{z}_s + d) - W_s(\tilde{z}_s).$$

Given $(\tilde{z}_b, \tilde{z}_s)$, the bargaining problem is thus

$$\max_{q,d} S_b(q, d; \tilde{z}_b, \tilde{z}_s) + S_s(q, d; \tilde{z}_b, \tilde{z}_s), \tag{6}$$

subject to the bargaining rule

$$S_b(q, d; \tilde{z}_b, \tilde{z}_s) = \theta[S_b(q, d; \tilde{z}_b, \tilde{z}_s) + S_s(q, d; \tilde{z}_b, \tilde{z}_s)],$$

and the liquidity constraint

$$d \leq \tilde{z}_b.$$

Using the result that $W_b(z)$ and $W_s(z)$ are linear, the bargaining problem (6) can be reformulated as

$$\max_{q, d \leq \tilde{z}_b} \{U(q) - C(q)\}, \text{ s.t.} \tag{7}$$

$$(1-\theta)[U(q) - d] = \theta[-C(q) + d].$$

Note that the bargaining solution q, d depend only on buyer's balance $\tilde{z}_b$, confirming our conjecture.

Symmetric Stationary Monetary Equilibrium

Define symmetric stationary monetary equilibrium as follows:

Definition 1 *A symmetric stationary monetary equilibrium consists of the price system* $\{\phi_t\}_{t=0}^{\infty}$*, the allocation* (q, d, z_b, z_s) *and the policy* $\{M_t, T_t\}_{t=0}^{\infty}, \mu$*, such that*

a. *(buyer's optimization) given* $z_{b,0}$ *and* $\{\phi_t\}_{t=0}^{\infty}$*,* $z' = z_b$ *solves (3);*

b. *(seller's optimization) given* $z_{s,0}$ *and* $\{\phi_t\}_{t=0}^{\infty}$*,* $z' = z_s$ *solves (5);*

c. *(CM clears)* $\phi_t M_{t+1} = z_b + z_s$*;*

d. *(DM bargaining)* $q = q(z_b/\mu)$*,* $d = d(z_b/\mu)$ *solve (6) ;*

e. *(Issuer's budget constraint) given* ϕ_t*,* $\{M_t, \mu, T_t\}$ *satisfies (1);*

f. *(monetary, stationary)* $\phi_t > 0, \phi_t/\phi_{t+1} = \mu$*.*

Bargaining Solution

Define $D(q) \equiv (1-\theta) U(q) + \theta C(q)$. It is straightforward to show the following lemma, which characterizes the bargaining solution.

Lemma 1 *The bargaining solution* $\{q(\tilde{z}), d(\tilde{z})\}$ *satisfies*

$$\begin{aligned} d &= \min\{\tilde{z}, (1-\theta) U(q^*) + \theta C(q^*)\}, \\ q &= D^{-1}(d). \end{aligned}$$

Proof. Omitted. ■

Intuitively, when the buyer brings enough balance to finance the first best consumption (i.e. $\tilde{z} \geq D(q^*)$), then unconstrained trade is conducted with terms of trade given by q^* and $D(q^*)$. However, when the buyer is constrained (i.e. $\tilde{z} < D(q^*)$), then she spends all, $d = \tilde{z}$, to buy $q = D^{-1}(z) < q^*$.

Existence of Monetary Equilibrium

Since $W_s(z)$ and $V_s(z)$ are linear, if a seller buys money in the period t CM and resells it in the $t+1$ CM, the rate of return in terms of utility is $\beta\phi_{t+1}/\phi_t - 1 = \beta/\mu - 1$. Therefore, whenever $\mu < \beta$, the seller's money demand z_s is infinite, and hence the money market cannot be cleared. On the other hand, when $\mu > \beta$, we must have $z_s = 0$. Intuitively, sellers have no need to spend money in the DM, and thus they have no incentive to buy money in the CM as long as its rate of return is negative (i.e. $\mu > \beta$). Moreover, $z_s = 0$ for $\mu \searrow \beta$. Similarly, a buyer will choose to bring an infinite amount for the next CM when $\mu < \beta$ and choose to bring zero balance to the next CM when $\mu > \beta$ (but may still bring balance for spending in the DM). In other words, the cash in advance constraint, $d \leq \tilde{z}_b$, is always binding in the DM when $\mu > \beta$. In this case, using Lemma (1) and ignoring the constant terms, we can rewrite the buyer's optimization problem (3) in the CM as

$$\max_q \{-\mu D(q) + \beta U(q)\}, \text{ s.t. } q \leq q^*. \tag{8}$$

Intuitively, a buyer chooses q in the DM. The benefit from consumption is $\beta U(q)$ and the cost of acquiring the balance for trade is $\mu D(q)$. A buyer will choose $q = z_b = 0$ when $\beta U'(q) - \mu D'(q) < 0$ for $q \to 0$. That is, the marginal incentive to bring money for DM consumption is negative even when $q \to 0$. This condition can be simplified to

$$\lim_{q \to 0} [\beta - \mu(1-\theta)] U'(q) < 0,$$

which is satisfied when $\mu > \overline{\mu} \equiv \beta/(1-\theta)$. Therefore, $z_b = q = 0$ and the monetary equilibrium does not exist when $\mu > \overline{\mu}$. For a buyer, since the opportunity cost of carrying nominal balances is increasing in the money growth rate μ, and the return from carrying balances for trade is also increasing in the bargaining weight θ, she has no incentives to hold money when μ is too high or θ is too low. These are the two inefficiencies highlighted in the monetary literature: cash-in-advance constraint and holdup problem. The following proposition characterizes the equilibrium.

Proposition 1 *A monetary equilibrium exists iff $\mu \in [\beta, \overline{\mu}]$. If $\mu > \beta$, then $q < q^*$; if $\mu \to \beta$, then $q = q^*$.*

According to this proposition, the first best allocation with $q = q^*$ cannot be supported when $\mu > \beta$. The idea is that, to consume q^* in the next DM, a buyer needs to bring $\mu d^* = \mu[(1-\theta) U(q^*) + \theta C(q^*)]$ money balance in the CM. So the marginal utility gain w.r.t. q is $\beta U'(q^*)$ while the marginal cost of acquiring the balance is $\mu[(1-\theta) U'(q^*) + \theta C'(q^*)] = \mu U'(q^*)$. As a result, a buyer has an incentive to marginally reduce q below q^* when

$$(\beta - \mu) U'(q^*) < 0, \tag{9}$$

which is true whenever $\beta < \mu$.

So deflating the economy at the discount rate is necessary and sufficient for implementing the first best allocation. Furthermore, the money-issuer's budget constraint implies that a lump-sum tax, $T_t = (\beta - 1) M_t \phi_t < 0$, is needed to implement the first best. If the money issuer has no taxation power, then this simple lump-sum transfer scheme cannot implement the first best. The natural question is: can the first best be supported by using more general transfer schemes? It calls for a mechanism design approach for examining general transfer functions.

Summary

In this section, we learn that in a monetary economy with lump sum transfers:

1. A monetary equilibrium does not exist when money growth is high, or buyer's bargaining power is low;

2. Without tax authority, first-best allocation can never be achieved.

3 Optimal Money Mechanism

Suppose the money issuer can conduct an intervention at night *after* the CM is closed. We use a mechanism design approach to design the optimal intervention and interpret the money issuer as the mechanism designer. We consider the following information structure: the money issuer can distinguish between buyers and sellers, but cannot observe an agent's past actions, nor the money balances he brings from the CM. The relevant space of agent types is thus two dimensional: whether she is a buyer or a seller, and how much money she holds. Thanks to the revelation principle, any equilibrium allocation of a Bayesian game under a mechanism can be implemented by a direct mechanism, where agents report their private information of type to the mechanism designer (here the money issuer), and the mechanism designer makes the money transfer based on the type reported.

Let us first briefly describe the basic idea of the mechanism and then give the formal setup below. Since the type of being a buyer or a seller is perfectly observable, agents only need to report their money holding to the money issuer. Specifically, if an agent leaves the CM market with $\bar{z}$ and decides to skip the intervention, then he will end this subperiod with exactly $\bar{z}$. But if this agent plans to participate in the intervention, then he will need to report his balance to the issuer. Note that it is feasible for an agent with money balances $\bar{z}$ to report an amount below $\bar{z}$ (i.e. hiding money) but infeasible to report an amount above $\bar{z}$ because over-reporting can be verified. Given the reported money holding, the money issuer will ask the agent to pay B (to receive if negative) in terms of money balances, as a function of the agent's type and the report. So after the intervention, the end of period money holding of this agent is $z = \bar{z} + B$, where z is the post transfer balance and $\bar{z}$ is the pre-transfer balance. Since the transfer B is known (and indeed set by the mechanism), there is no difference between asking the agent to report the pre-transfer $\bar{z}$ or the post-transfer z. For notational convenience, we will assume the report is about the post transfer balance z and the pre-transfer balance can be inferred directly.

Formally, an agent participating in the mechanism needs to give report $\widehat{z}$ of the actual (post transfer) balance z, subject to $\widehat{z} \leq z$. A money mechanism $\mathcal{M} \equiv \{B_b(\widehat{z}), B_s(\widehat{z}), \mu\}$ consists of transfer functions for buyers, $B_b(\widehat{z})$, and for sellers, $B_s(\widehat{z})$, and a money growth rate μ.

CM and DM decision

A type $j = b, s$ agent's DM value function under a money mechanism $\mathcal{M}$ remains the same, given by (2) and (4). A type j's CM value function becomes

$$W_j(z) = z + \max_{z', e \in \{0,1\}} \left\{ -z' + eJ_j(z') + (1-e)\beta V_j\left(\frac{z'}{\mu}\right) \right\}, \tag{10}$$

where $e = 1$ and $e = 0$ denote respectively the decision to participate and to not participate in the mechanism. $J_j(z)$ denotes the continuation value when $e = 1$, given by

$$J_j(z) = \max_{\widehat{z}} \left\{ -B_j(\widehat{z}) + \beta V_j(\frac{z}{\mu}) \right\}, \text{ s.t.}$$

$$\widehat{z} \le z.$$

Here, an agent with z chooses to report $\widehat{z}$ subject to the constraint that over-reporting is not feasible, and his report will result a payment $B_j(\widehat{z})$.

Incentive-compatibility for buyers

It is straightforward to establish that the bargaining solution $\{q(z), d(z)\}$ under a money mechanism $\mathcal{M}$ is still characterized by Lemma (1). Using the linearity of $W_b(z)$ and ignoring the constant terms, one can reformulate the buyer's CM problem under mechanism $\mathcal{M}$ as

$$\max_{e\in\{0,1\},\widehat{z},q} \{e[-B_b(\widehat{z}) - \mu D(q) + \beta U(q)] + (1-e)[-\mu D(q) + \beta U(q)]\}, \text{ s.t.} \tag{11}$$

$$\widehat{z} \le \mu D(q). \tag{12}$$

where the first term in the objective function captures the payoff of participating in the mechanism, and the second term captures the payoff of skipping the mechanism. Here, if a buyer decides not to participate in the mechanism, he just brings $z = \mu D(q)$ from the CM, and that will allow him to consume q in the following DM. If a buyer decides to participate and intends to have a post-transfer balance of $z = \mu D(q)$, he needs to bring $\mu D(q) + B_b(\widehat{z})$ from the CM, report $\widehat{z} \le \mu D(q)$, and this will allow him to consume q in the following DM.

Definition 2 *An allocation (q, d, z_b, z_s) is incentive compatible for buyers under a money mechanism $\mathcal{M}$ if $e = 1, \widehat{z} = z_b$, and $q = D^{-1}(d) = D^{-1}[z_b/\mu]$ solve (11).*

To induce buyers to participate the mechanism (i.e. $e = 1$), it is necessary to have an incentive-compatible allocation (q, d, z_b, z_s) satisfying

$$-B_b(z_b) - \mu D(q) + \beta U(q) \ge \max_{q'} \{-\mu D(q') + \beta U(q')\}. \tag{13}$$

Here, the LHS captures the payoff for participating in the mechanism, and the RHS captures the payoff for skipping it.

Incentive-compatibility for sellers

Similarly, using the linearity of $W_s(z)$, and ignoring the constant terms, one can reformulate the seller's problem in the CM as

$$\max_{e\in\{0,1\},\widehat{z},z}\left\{-z'+e[-B_s(\widehat{z})+\frac{\beta}{\mu}z]\right\}, \text{ s.t. } \widehat{z}\leq z. \tag{14}$$

Here, a seller not participating in the mechanism has no reason to bring money and thus the additional payoff is zero. If a seller decides to participate and intends to have a post-transfer balance of z, he needs to bring $z+B_b(\widehat{z})$ from the CM, report $\widehat{z}\leq z$, and this balance will have a continuation value $\beta z/\mu$.

Definition 3 *An allocation (q,d,z_b,z_s) is incentive compatible for sellers under a money mechanism $\mathcal{M}$ if $e=1$ and $\widehat{z}=z=z_s$ solve (14).*

Notice the value of choosing $e=0$ is $\max_{\widehat{z},z'}\left\{-z'+e\left[-B_s(\widehat{z})+\frac{\beta}{\mu}[z'+T_s(\widehat{z})]\right]\right\}=0$. So to induce sellers to participate the mechanism, it is necessary to have an incentive-compatible allocation (q,d,z_b,z_s) which satisfies

$$-z_s-B_s(z_s)+\frac{\beta}{\mu}z_s\geq 0. \tag{15}$$

Here, the LHS captures the payoff for participating in the mechanism, and the RHS captures the payoff for skipping it.

Money issuer's budget constraint

A money issuer has to balance its budget, or self-financed:

Definition 4 *A money mechanism $\mathcal{M}\equiv\{B_b(z),B_s(z),\mu\}$ is self-financed under the allocation (q,d,z_b,z_s) if*

$$-B_b(z_b)-B_s(z_s)=(1-1/\mu)(z_b+z_s). \tag{16}$$

This budget constraint states that the issuer's total expenditure on transfers (LHS) has to be financed by money creation (RHS). Since the total real balances are z_b+z_s at the end of a period, and $(z_b+z_s)/\mu$ at the beginning of a period. The RHS denotes the total balances created within a period.

Notice that if an allocation (q,d,z_b,z_s) is incentive compatible for buyers and sellers under a mechanism $\mathcal{M}$, then the equilibrium conditions a, b, d, and f in the definition are satisfied. Furthermore, if $\mathcal{M}$ is self-financed under (q,d,z_b,z_s), then the equilibrium conditions c and e in the definition are satisfied. So any incentive-compatible allocation can be implemented as the equilibrium allocation by a mechanism if it is self-financed. In particular, we are interested in whether the first-best allocation can be implemented. The following definition introduces this concept formally.

Implementability of first best

Definition 5 *A money mechanism $\mathcal{M}$ implements the first best if*

a. there exists (z_b, z_s) such that the first-best allocation (q^*, d^*, z_b, z_s) is incentive compatible for buyers and sellers; and

b. $\mathcal{M}$ is self-financed under the first-best allocation (q^*, d^*, z_b, z_s).

Define a threshold level $\overline{\theta}$ of the buyer's bargaining power given by

$$\overline{\theta} \equiv \frac{1-\beta}{1-\frac{C(q^*)}{U(q^*)}}.$$

Note that $\overline{\theta} \in (0,1)$ iff

$$\beta U(q^*) > C(q^*).$$

We are going to assume this condition holds throughout the paper. Intuitively, for a buyer considering whether or not to bring cash from CM to finance a first-best trade in the DM next period, $\beta U(q^*)$ is the maximum (discounted) utility gain from this trade while $C(q^*)$ is the minimum price needed to induce the seller to trade. When the above condition is violated, there is no hope for first-best trade in a monetary economy in which agents need to bring cash to trade.

The following proposition characterizes the implementability of the first best allocation under an optimally designed money mechanism.

Proposition 2 *There exists a money mechanism $\mathcal{M}$ which implements the first best if and only if $\theta \geq \overline{\theta}$.*

Proof. (Sketch) We sketch the proof for Proposition (2) as follows. First, we show that if $\theta < \overline{\theta}$ then there does not exist any money mechanism $\mathcal{M}$ which implements the first best. Suppose not, and denote (q^*, d^*, z_b, z_s) as the first-best allocation implemented. Since the equilibrium exists as the first-best allocation, we must have $\mu \geq \beta$. Denote $\eta \equiv -\left(1-\frac{\beta}{\mu}\right) z_s - B_s(z_s)$. Since (q^*, d^*, z_b, z_s) is incentive compatible for sellers, from (15) we have $\eta \geq 0$. The fact that (q^*, d^*, z_b, z_s) is incentive compatible for buyers implies that $\mu d^* = z_b$. Substituting (16) and $\mu d^* = z_b$ into the definition of η, we have

$$-B_b(z_b) - \mu d^* = -\eta - \frac{1-\beta}{\mu} z_s - d^* \tag{17}$$

Since (q^*, d^*, z_b, z_s) is also incentive compatible for buyers, from (13) we have

$$\begin{aligned} & \max_{q'} \{-\mu D(q') + \beta U(q')\} \\ \leq \; & -B_b(z_b) - \mu d^* + \beta U(q^*), \\ = \; & \underbrace{-\eta - \frac{1-\beta}{\mu} z_s}_{\leq 0} + \left(\theta - \overline{\theta}\right)[U(q^*) - C(q^*)], \end{aligned}$$

where we have substituted (17) and used the fact that $\beta U(q^*) - d^* = (\theta - \overline{\theta})[U(q^*) - C(q^*)] < 0$. Since we have $\max_{q'}\{-\mu D(q') + \beta U(q')\} \geq 0$, there is contradiction.

On the other hand, if $\theta \geq \overline{\theta}$, we can construct a money mechanism $\mathcal{M}$ which implements the first best. Consider the following money mechanism: $B_s(z) = 0$ for all z, $\mu = \overline{\mu}$ and

$$B_b(z) = \begin{cases} (1-\overline{\mu})\, d^*, \text{ if } z = \overline{\mu} d^* \\ 0, \text{ otherwise.} \end{cases}$$

■

Proposition 1 shows that, without the authority to enforce taxation, the first best allocation cannot be achieved by simple lump sum transfers. Recall that buyers have incentives to marginally reduce q below q^* when (9) is satisfied (i.e. $\beta < \mu$). According to Proposition 2, a suitably designed mechanism can still implement the first best. To do that, the transfer scheme $B_b(z)$ has to be designed to induce buyers to carry the right amount of money balances to finance the first-best trade. In particular, when the transfer scheme is non-linear and optimally designed, a buyer no longer has the marginal incentive to reduce q below q^* even when $\beta < \mu$. To give a concrete example, a mechanism may make a big transfer to buyers who bring and report a sufficiently high money balance, and make no transfers to buyers who bring and report too little balances. In an equilibrium in which all buyers cooperate and receive big transfers, the inflation is high. Hence, a deviator who brings too little money and receives no transfers will suffer a loss in purchasing power. Under this non-linear scheme, a buyer does not want to lower his money holding too much because that will significantly reduce his surplus from DM trades. This explains why the first best can be supported by the optimal mechanism. However, the power of this scheme is limited by the size of buyer's DM trade surplus, which in turn depends on θ. That explains why the first best can no longer be supported when θ is too low.

Characterization of optimal mechanism

The above discussion suggests that, to support the first best, transfers to buyers is needed, and hence money growth is positive. The following proposition formally establishes this finding which characterizes all optimal money mechanisms.

Proposition 3 *If a money mechanism $\mathcal{M} \equiv \{B_b(z), B_s(z), \mu\}$ implements the first best, then $\mu > 1$.*

Proof. Suppose there exists a mechanism $\mathcal{M} \equiv \{B_b(z), B_s(z), \mu\}$ that implements the first best with

$\mu \leq 1$. Then from the proof of Proposition 2, we have

$$\begin{aligned} -D(q^*) + \beta U(q^*) &\geq -B_b(z_b) - \mu d^* + \beta U(q^*), \\ &\geq \max_{q'} \{-\mu D(q') + \beta U(q')\}, \\ &\geq \max_{q'} \{-D(q') + \beta U(q')\}, \end{aligned}$$

which is contradiction since $\beta < 1$. ■

Simple examples

To illustrate the basic idea, we propose a simple example of a direct mechanism, and then an example of an indirect mechanism.

(i) Direct mechanism

This simple mechanism makes no transfers to sellers, so $B_s(z) = 0$ for all z. The money growth is set to $\mu = \overline{\mu}$ and the money created is used to finance transfers to buyers such that

$$B_b(z) = \begin{cases} -(\overline{\mu} - 1) d^*, \text{ if } z = \overline{\mu} d^* \\ 0, \text{ otherwise.} \end{cases}.$$

First, note that $\overline{\mu} > 1$ when $\theta \geq \overline{\theta}$, so that a buyer with $z = \overline{\mu} d^*$ can receive a positive transfer $(\overline{\mu} - 1) d^* > 0$ from the money issuer. As a result, this buyer has a payoff which equals to $-d^* + \beta U(q^*)$, which is positive iff $\theta \geq \overline{\theta}$. Under this mechanism, a buyer with $z \neq \overline{\mu} d^*$ does not receive transfers, and thus has non-positive payoff because $\mu = \overline{\mu}$. Notice that this scheme is non-linear with respect to the buyer's money holding z.

(ii) Indirect mechanism: fixed fee and interest payments

Following Andolfatto (2010), we derive in Appendix A an indirect mechanism with the following features: the money issuer imposes a fixed fee B on buyers who can then collect interest on their money balances at the rate R in the end of the CM:

$$\begin{aligned} R &= \frac{\mu}{\beta} - 1, \\ B &= \mu (1 - \beta) d^*. \end{aligned}$$

We show that, for sufficiently high μ, the first best allocation can be supported if $\theta \geq \overline{\theta}$. The basic idea is that the interest payment offsets the buyers' opportunity cost of carrying money balances to the DM. This interest payment is financed by the fixed fee paid by the buyers. In order to induce them to pay this fee, the monetary growth has to be sufficiently high so that non-participants' trade surplus in the DM is sufficiently low. Notice that this scheme is piece-wise linear: a fixed fee plus a linear transfer with respect to the buyer's money holding z.

Summary

In this section, we learn that under an optimal monetary mechanism:

1. First best allocation can be achieved when buyer's bargaining power is not too low;

2. Non-linear transfer scheme, and monetary expansion are needed;

3. First-best can be implemented by a simple indirect mechanism with interest on buyers' balances fixed fee, financed by fixed fees and monetary expansion.

4 Electronic money with limited participation

Suppose there is an e-money issuer who maintains the supply of another medium of exchange, *e-money*, in additional to money. E-money shares all the basic properties of money: divisible, durable, portable and cannot be counterfeited. However, e-money is different from money with one of the features detailed in this and the following section. Like the money issuer in the previous section, the e-money issuer can distinguish between buyers and sellers, but cannot observe agent's portfolio of money z nor e-money n. Agents thus report their portfolio to the e-money issuer, and the e-money issuer makes the e-money transfer based on the report. In this section, we are interested in an economy where the money mechanism is exogenously given by a money growth rate $\mu \geq 1$, and the corresponding lump sum transfer of money $T = (1 - \mu^{-1})(z_b + z_n) \geq 0$. As a result, there is no strategic interaction between the money issuer and the e-money issuer. Also, assume the e-money issuer has to maintain a constant exchange rate between money and e-money in the CM. As a result, μ is also the growth rate of e-money. This is to capture the fact that real word e-money products often involves this feature (e.g. denominated at par). Note that removing this restriction will only make the e-money issuer more powerful, strengthening our conclusion. Finally, one should note that, just like the money issuer, the e-money issuer dose not know the amount of e-money transferred nor the identities of the payer and payee in the DM. Otherwise the first best may be trivially implemented.

E-money mechanism

As discussed in the Introduction, while anyone can freely participate in a monetary payment system, an e-money issuer is able to restrict participation in an e-money payment system. Specifically, the e-money issuer has the power to exclude agents from holding e-money in the DM. Notice that the e-money issuer still does not know nor control how e-money are used by agents.

Again, suppose the e-money issuer conducts an intervention at the end of the CM. We use mechanism design to characterize the optimal mechanism. Let $e_j = 1$ indicates that a type $j = b, s$ agent chooses to participate in the e-money mechanism. Any agent choosing $e_j = 0$ will be excluded from the e-money system in the next DM: cannot hold, pay or receive e-money in the next DM. Since agents are anonymous, this penalty can last only one period. An e-money mechanism $\mathcal{M}_E$ consists of two e-money transfer functions based on the portfolio reported, denoted as $\mathcal{M}_E \equiv \{B_b(z,n), B_s(z,n)\}$.

CM and DM decision

In the DM, the buyer's value function under an e-money mechanism $\mathcal{M}_E$, $V_b(z,n,e_b)$, is

$$\begin{aligned} V_b(z,n,e_b) &= e_b e_s \{U[q(z+n)] + W_b[z+n-d(z+n)]\} \\ &\quad + (1-e_s e_b)\{U[q(z)] + W_b[z+n-d(z)]\}. \end{aligned} \tag{18}$$

Here, the continuation value in the CM, $W_b(a)$, depends only the sum of the real balances of money and e-money $a \equiv z+n$. The first term captures the case when both buyers and sellers participate in the e-money system (i.e. $e_b = e_s = 1$). In this case, the buyer can use the total real balances a to finance the trade. Otherwise (i.e. $e_b e_s = 0$), then the buyer can only use the money z to finance the trade. The CM value function of the buyer under an e-money mechanism $\mathcal{M}_E$ is given by

$$W_b(a) = \max_{z',n',e_b\in\{0,1\}} \left\{ a - z' - n' + e_b J_b(z',n') + (1-e_b)\beta V_b\left(\frac{z'}{\mu}, \frac{n'}{\mu}, 0\right)\right\}, \text{ s.t.} \tag{19}$$

where $J_b(z,n)$ is the buyer's value function if she chooses to participate in the mechanism, which is given by

$$J_b(z,n) = \max_{\hat{z},\hat{n}} \left[-B_b(\hat{z},\hat{n}) + \beta V_b\left(\frac{z}{\mu}, \frac{n}{\mu}, 1\right)\right]$$

$$\begin{aligned} \hat{z} &\le z, \\ \hat{n} &\le n. \end{aligned}$$

In the DM, the value function, $V_s(a,e_s)$, of a seller who has joined an e-money mechanism $\mathcal{M}_E$, is

$$\begin{aligned} V_s(a,e_s) &= e_s e_b \{-C[q(z_b+n_b)] + W_s[a+d(z_b+n_b)]\} \\ &\quad (1-e_s e_b)\{-C[q(z_b)] + W_s[a+d(z_b)]\}. \end{aligned} \tag{20}$$

Similarly, if $e_b e_s = 1$, then the buyer can use $z_b + n_b$ to finance the trade. If $e_b e_s = 0$, then the buyer can only use z_b to finance the trade. In the CM, the seller's value function under an e-money mechanism $\mathcal{M}_E$, $W_s(a)$, is

$$W_s(a) = \max_{z', n', e_s \in \{0,1\}} \left\{ a - z' - n' + e_s J_s(z', n') + (1 - e_s) \beta V_s \left(\frac{z' + n'}{\mu}, 0 \right) \right\}, \text{ s.t.} \tag{21}$$

$$\begin{aligned} \widehat{z} &\leq z', \\ \widehat{n} &\leq n'. \end{aligned}$$

where $J_s(z, n)$ is the CM value of the seller if she chooses to participate in the mechanism, which is given by

$$J_s(z, n) = \max_{\widehat{z}, \widehat{n}} \left\{ -B_s(\widehat{z}, \widehat{n}) + \beta V_s \left(\frac{z + n}{\mu}, 1 \right) \right\}$$

$$\begin{aligned} \widehat{z} &\leq z, \\ \widehat{n} &\leq n. \end{aligned}$$

Incentive-compatibility for buyers

It is straightforward to establish that the bargaining solution $\{q(a), d(a)\}$ under a money mechanism $\mathcal{M}$ is still characterized by Lemma (1). Using the linearity of $W_b(a)$, and ignoring the constant terms, one can reformulate the buyer's problem in the CM under $e_s = 1$ as

$$\max_{e_b \in \{0,1\}, z', \widehat{z}, \widehat{n}, q,} \{ e_b [-\mu D(q) - B_b(\widehat{z}, \widehat{n}) + \beta U(q)] + (1 - e_b) [-\mu D(q) + \beta U(q)] \}, \text{ s.t.} \tag{22}$$

$$\begin{aligned} \widehat{z} &\leq z' \\ \widehat{n} &\leq \mu D(q) - z'. \end{aligned}$$

As before, if $e_b = 0$, the buyer brings $\mu D(q)$ money balances to buy q in the DM. If the buyer chooses $e_b = 1$ and intends to have a post-transfer portfolio of (z, n), he needs to obtain total balances $\mu D(q) + B_b(\widehat{z}, \widehat{n})$ from the CM, report $\widehat{z}$ and $\widehat{n}$, and them buy q in the following DM.

Definition 6 *An allocation $(q, d, z_b, z_s, n_b, n_s)$ is incentive compatible for buyers under an e-money mechanism $\mathcal{M}_E$ if $e_b = 1$, $\widehat{z} = z' = z_b$, $\widehat{n} = n_b$ and $q = D^{-1}(d) = D^{-1}[(z_b + n_b)/\mu]$ solve (22) given $e_s = 1$.*

To induce buyers to participate the mechanism, it is necessary to have an incentive-compatible allocation $(q, d, z_b, z_s, n_b, n_s)$ satisfying

$$-B_b(z_b, n_b) - \mu D(q) + \beta U(q) \geq \max_{q'} \{-\mu D(q') + \beta U(q')\}. \tag{23}$$

Here, the LHS captures the payoff for joining the e-money mechanism, and the RHS captures the payoff for skipping it.

Incentive-compatibility for sellers

Similarly, using the linearity of $W_s(a)$, and ignoring the constant terms, one can reformulate the seller's problem in the CM under $e_b = 1$ as

$$\max_{\substack{e_s \in \{0,1\}, \\ \widehat{z}, z', \widehat{n}, n'}} \left\{ \begin{array}{c} -z' - n' + (1 - e_s)\beta \left[\frac{z'+n'}{\mu} + d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] \\ +e_s \left[-B_s(\widehat{z}, \widehat{n}) + \beta \left[\frac{z'+n'}{\mu} + d\left(\frac{z_b+n_b}{\mu}\right) - C\left[q\left(\frac{z_b+n_b}{\mu}\right)\right]\right]\right] \end{array} \right\}, \text{ s.t.} \tag{24}$$

$$\begin{aligned} \widehat{z} &\leq z', \\ \widehat{n} &\leq n'. \end{aligned}$$

Again, a seller joining the e-money mechanism (i.e. $e_s = 1$) has to bring extra balances to pay for the transfer $B_s(\widehat{z}, \widehat{n})$.

Definition 7 *An allocation $(q, d, z_b, z_s, n_b, n_s)$ is incentive compatible for sellers under an e-money mechanism $\mathcal{M}_E$ if $e_s = 1$, $\widehat{z} = z' = z_s$, $\widehat{n} = n' = n_s$ solve (24) under $e_b = 1$.*

To induce sellers to participate in the mechanism, it is necessary to have an incentive-compatible allocation $(q, d, z_b, z_s, n_b, n_s)$ satisfying

$$-B_s(z_s, n_s) - \left(1 - \frac{\beta}{\mu}\right)(z_s + n_s) + \beta[d - C(q)] \geq \beta \left\{d\left(\frac{z_b}{\mu}\right) + C\left[q\left(\frac{z_b}{\mu}\right)\right]\right\}, \tag{25}$$

where the LHS captures the payoff for participating in the e-money mechanism, and the RHS captures the payoff for skipping it.

E-money issuer's budget constraint

An e-money issuer has to balance its budget, or self-financed:

Definition 8 *An e-money mechanism* $\mathcal{M}_E = \{B_b(z,n), B_s(z,n)\}$ *is self-financed with limited participation under the allocation* $(q, d, z_b, z_s, n_b, n_s)$ *if*

$$-B_b(z_b, n_b) - B_s(z_s, n_s) = \left(1 - \frac{1}{\mu}\right)(n_b + n_s). \tag{26}$$

Here, the e-money issuer finances transfers to buyers and sellers by issuing e-money balances.

Implementability of first best

Definition 9 *An e-money mechanism* $\mathcal{M}_E$ *implements the first best with limited participation if*

a. there exists (z_b, z_s, n_b, n_s) *such that the first-best allocation* $(q^*, d^*, z_b, z_s, n_b, n_s)$ *is incentive compatible for buyers and sellers; and*

b. $\mathcal{M}_E$ *is self-financed with limited participation under the first-best allocation* $(q^*, d^*, z_b, z_s, n_b, n_s)$.

Define a threshold function in terms of θ and μ:

$$\Theta(\theta, \mu) \equiv \frac{\overline{\theta} - \beta}{1 - \beta} + \frac{\max_q \{\beta U(q) - \mu D(q)\}}{(1 - \beta)[U(q^*) - C(q^*)]}$$

The following proposition establishes the condition under which the first best can be achieved by an optimal e-money mechanism with limited participation.

Proposition 4 *Suppose* $\mu \geq 1$*, then there exists an e-money mechanism* $\mathcal{M}_E$ *that implements the first best with limited participation if and only if* $\theta \geq \Theta(\theta, \mu)$.

Proof. First, we want to show that if $\theta < \Theta(\theta, \mu)$ then there does not exist an e-money mechanism $\mathcal{M}_E$ that implements the first best with limited participation. Suppose not, then there exists an e-money mechanism $\mathcal{M}_E$ that implements a first-best $(q^*, d^*, z_b, z_s, n_b, n_s)$. Since the equilibrium exists as the first-best allocation, we must have $\mu \geq \beta$. Define

$$\begin{aligned} \eta \equiv & -\left(1 - \frac{\beta}{\mu}\right)(z_s + n_s) - B_s(z_s, n_s) \\ & + \beta\left[d^* - C(q^*) - d\left(\frac{z_b}{\mu}\right) + C\left[q\left(\frac{z_b}{\mu}\right)\right]\right]. \end{aligned}$$

The fact that $(q^*, d^*, z_b, z_s, n_b, n_s)$ is incentive compatible for sellers and buyers implies that $\eta \geq 0$ and $\mu d^* = z_b + n_b$. Substituting (26) and $\mu d^* = z_b + n_b$ to (25), we have

$$-z_b - n_b - B_b(z_b, n_b) = -\eta - A - \beta C(q^*) - (1 - \beta) d^*, \tag{27}$$

where

$$A \equiv \beta \left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] + \left(1-\frac{\beta}{\mu}\right) z_s + \left(1-\frac{1}{\mu}\right) z_b + \left(\frac{1-\beta}{\mu}\right) n_s \geq 0$$

Notice that the definition of $\Theta(\theta,\mu)$ implies that $\theta < \Theta(\theta,\mu)$ if and only if

$$\left[\beta(1-\theta)+\theta-\overline{\theta}\right]\left[U(q^*)-C(q^*)\right] < \max_q \left\{-\mu D(q)+\beta U(q)\right\} \tag{28}$$

Since $(q^*, d^*, z_b, z_s, n_b, n_s)$ is also incentive compatible for buyers, from (23) we have

$$\begin{aligned}
& \max_q \left\{-\mu D(q)+\beta U(q)\right\} \\
\leq \; & -\mu d^* - B_b(z_b, n_b) + \beta U(q^*) \\
= \; & -B_b(z_b, n_b) - z_b - n_b + \beta U(q^*) \\
= \; & -\eta - A - (1-\beta) d^* + \beta\left[U(q^*) - C(q^*)\right] \\
= \; & -\eta - A + \beta\left[d^* - C(q^*)\right] + \beta U(q^*) - d^* \\
= \; & -\eta - A + \left[\beta(1-\theta)+\theta-\overline{\theta}\right]\left[U(q^*)-C(q^*)\right] \\
< \; & -\eta - A + \max_q \left\{-\mu D(q)+\beta U(q)\right\}.
\end{aligned}$$

where we have substituted (27), (28) and used the fact that $d^* - C(q^*) = (1-\theta)\left[U(q^*)-C(q^*)\right]$. A contradiction.

On the other hand, if $\theta \geq \Theta(\theta,\mu)$, we can construct an e-money mechanism $\mathcal{M}_E$ that implements the first best with limited participation. Since $\theta \geq \Theta(\theta,\mu)$, we have $\varepsilon_0 \equiv -(1-\beta)d^* + \beta\left[U(q^*)-C(q^*)\right] - \max_q\left\{-\mu D(q)+\beta U(q)\right\} \geq 0$. Fix any $n_b > 0$ and $z_b > 0$ such that $n_b + z_b = \mu d^*$ and

$$\beta\left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] + \left(1-\frac{1}{\mu}\right) z_b \leq \varepsilon_0.$$

Consider the first-best allocation $(q^*, d^*, z_b, 0, n_b, 0)$ and the following e-money mechanism $\mathcal{M}_E$:

$$B_s(z,n) = \beta\left[d^* - C(q^*) - d\left(\frac{z_b}{\mu}\right) + C\left[q\left(\frac{z_b}{\mu}\right)\right]\right], \tag{29}$$

$$B_b(z,n) = \begin{cases} -B_s(z_n, n_n) - \left(1-\frac{1}{\mu}\right) n_b, \text{ if } z = z_b \text{ and } n = n_b \\ 0, \text{ otherwise} \end{cases}. \tag{30}$$

Then it is straightforward to verify that (29) implies (25) and (30) implies (26) under the first-best allocation $(q^*, d^*, z_b, 0, n_b, 0)$ and $\mathcal{M}_E$ constructed above. So $(q^*, d^*, z_b, 0, n_b, 0)$ is incentive compatible for sellers under$\mathcal{M}_E$, and $\mathcal{M}_E$ is self-financed with limited participation. Finally, substituting $\mu d^* = z_b + n_b$, and

(30) into $-\mu d^{*}-B_b(z_b,n_b)+\beta U(q^{*})-\max_q\{-\mu D(q)+\beta U(q)\}$, then we have

$$\begin{aligned}
& -\mu d^{*}-B_b(z_b,n_b)+\beta U(q^{*})-\max_q\{-\mu D(q)+\beta U(q)\} \\
= & -\mu d^{*}+\left(1-\frac{1}{\mu}\right)(z_b+n_b)+\beta\left[d^{*}-C(q^{*})\right]+\beta U(q^{*})+\max_q\{-\mu D(q)+\beta U(q)\} \\
& -\beta\left[d\left(\frac{z_b}{\mu}\right)-C\left[q\left(\frac{z_b}{\mu}\right)\right]\right]-\left(1-\frac{1}{\mu}\right)z_b \\
\geq & -(1-\beta)\,d^{*}+\beta\left[U(q^{*})-C(q^{*})\right]-\max_q\{-\mu D(q)+\beta U(q)\}-\varepsilon_0 \\
= & \ \varepsilon_0-\varepsilon_0=0,
\end{aligned}$$

thus (23) is satisfied given the first-best allocation $(q^{*},d^{*},z_b,0,n_b,0)$ and $\mathcal{M}_E$ constructed above. Thus $(q^{*},d^{*},z_b,0,n_b,0)$ is also incentive compatible for buyers under$\mathcal{M}_E$, and $\mathcal{M}_E$ implements the first best with limited participation. ■

This proposition shows that, to implement the first best using this e-money mechanism, buyers' bargaining power and inflation need to satisfy $\theta\geq\Theta(\theta,\mu)$. It is straightforward to show that $\Theta(\theta,\mu)$ is increasing in θ and decreasing in μ. An increase in μ facilitates the implementation of first best because it reduces the outside option of non-participants who use money only as their means of payments.[8] An increase in θ has two opposite effects. On the one hand, it helps achieve first best because the holdup problem is less severe and thus buyers have higher incentives to bring more e-money balances. On the other hand, it also increases the outside option of non-participants who also face less severe holdup problem when using money. But, in general, we know that $\Theta(\theta,\mu)\geq(\overline{\theta}-\beta)/(1-\beta)>0$. Therefore, the first best is implementable only if $\theta\geq(\overline{\theta}-\beta)/(1-\beta)$. We will discuss below why first best cannot be implemented for θ too low.

Essentiality of limited participation

Proposition 5 *If there exists a money mechanism $\mathcal{M}$ that implements the first best with μ, then there also exists an e-money mechanism $\mathcal{M}_E$ that implements the first best with limited participation under the same μ.*

Proof. Since $\mathcal{M}$ implements the first best, from the proof of Proposition 2 it is necessary to have

$$\begin{aligned}
0 \leq & -d^{*}+\beta U(q^{*})-\max_q\{-\mu D(q)+\beta U(q)\} \\
\leq & \ \beta\left[d^{*}-C(q^{*})\right]-d^{*}+\beta U(q^{*})-\max_q\{-\mu D(q)+\beta U(q)\} \\
= & \ \beta(1-\theta)\left[U(q^{*})-C(q^{*})\right]+\left(\theta-\overline{\theta}\right)\left[U(q^{*})-C(q^{*})\right]-\max_q\{-\mu D(q)+\beta U(q)\} \\
= & \ (1-\beta)\left[U(q^{*})-C(q^{*})\right]\left[\theta-\Theta(\theta,\mu)\right].
\end{aligned}$$

[8] Interesting, this is consistent with a popular view that inflation induces agents to adopt some e-money products. For example, Bitcoin is considered by some as safe haven from inflation.

Thus we have $\theta \geq \Theta(\theta, \mu)$, then from Proposition 4 there exists an e-money mechanism $\mathcal{M}_E$ that implements the first best. ■

This proposition implies that, fixing the money growth rate, an optimal e-money mechanism featuring limited participation is at least as good as an optimal money mechanism in implementing the first best allocation. This result may not hold in general when the e-money mechanism has to operate under a money growth rate different from that associated with the optimal money mechanism. Define

$$\underline{\theta} \equiv \frac{\overline{\theta} - \beta}{1 - \beta},$$

the following proposition gives conditions under which the e-money mechanism can out-perform a money mechanism.

Proposition 6 (Essentiality of e-money with limited participation) *If $\theta \in [\underline{\theta}, \overline{\theta})$, then first best allocation*

(i) cannot be implemented by any money mechanism;

(ii) can be implemented by an e-money mechanism with limited participation when $\mu \geq \bar{\mu}$.

Proof. Omitted here. ■

This proposition establishes the essentiality of e-money featuring limited participation. Part (i), implied by Proposition 2, states that no money mechanism can implement the first-best when $\theta < \overline{\theta}$. At the risk of being repetitive, we reproduce here the intuition: a money mechanism uses a non-linear transfer scheme to induce buyers to "cooperate" and to carry sufficient money balances. This scheme relies on the "punishment" of eroding deviating buyers' DM trade surplus by not giving them a transfer. However, the power of this scheme is limited by the size of buyers' trade surplus which depends on θ. When $\theta < \overline{\theta}$, buyer's trade surplus is insufficient for inducing them to carry the right money balances. In the extreme case of $\theta \to 0$, buyers have no surplus to be extracted.

Now, the ability of e-money issuer to limit participation provides an additional tool. By threatening to exclude agents from the participation in the e-money system, the issuer can now extract extra resources (especially from sellers), and can then use these extra resources to induce buyers to bring the right money balances. How much resources can be extracted from buyers and sellers? That is equal to the difference between the trade surplus of a e-money user and that of a money user. The power of this scheme is maximized when money users' trade surplus is zero, and this will happen when $\mu \geq \bar{\mu}$ (from Proposition 1). In this case the threat to exclude deviators allow the issuer to extract the whole of the (discounted) trade surplus, which equals to $\beta[U(q^*) - C(q^*)]$. This explains why e-money featuring limited participation is essential.

However, the power of this scheme is still insufficient to achieve first best when θ is too low. To illustrate by an extreme example with $\theta \to 0$. In this case, the buyer has no bargaining power and thus the price for

q^* is $d^* = U(q^*)$. So to induce buyers to bring d^* in the previous CM, the transfer B_b has to be sufficiently negative so that they still have a positive payoff:

$$-d^* - B_b + \beta U(q^*) \geq 0.$$

However, the above discussion implies that the maximum transfer the e-money issuer can make is $-B_b = -\beta[U(q^*) - C(q^*)]$, which is the whole of the (discounted) trade surplus. Plugging the values of d^* and B_b into the LHS, buyers' payoff becomes

$$-U(q^*) + \beta U(q^*) - \beta C(q^*) + \beta U(q^*)$$

As $\beta U(q^*) - C(q^*) \searrow 0$, their payoff becomes

$$-(1-\beta)[U(q^*) - C(q^*)] < 0.$$

Therefore, this example illustrates that, when θ is small and $\beta U(q^*) - C(q^*)$ is small (but remains positive, as assumed), the first best is not implementable by any e-money mechanism with limited participation. This explains part (ii) of the above proposition.

Overall, Proposition 2 and 6 characterize the implementability of first-best using money mechanism and e-money mechanism. When buyers' bargaining power is high ($\theta \geq \overline{\theta}$), an optimal money mechanism can implement the first best. Hence, e-money is not essential relative to money in this region. When buyers' bargaining power is moderate ($\overline{\theta} > \theta \geq \underline{\theta}$), only e-money featuring limited participation can implement the first best, given sufficiently high money growth rate ($\mu \geq \bar{\mu}$). Hence, e-money is essential relative to money in this region. Finally, when buyers' bargaining power is too low ($\underline{\theta} > \theta$), neither money nor e-money featuring limited participation can implement the first best.[9]

Characterization of optimal e-money mechanism

After establishing the essentiality of e-money, we now characterize the optimal e-money mechanism.

Proposition 7 *Suppose $\theta < \overline{\theta}$ (i.e., first best not implementable by any money mechanism). If there exists an e-money mechanism $\mathcal{M}_E = \{B_b(z,n), B_s(z,n)\}$ that implements the first best with limited participation, then $B_s(z_s, n_s) > 0$ and $B_b(z_b, n_b) < 0$.*

Proof. Suppose not, i.e., there does not exist any money mechanism but an e-money mechanism $\mathcal{M}_E = \{B_b(z,n), B_s(z,n)\}$ that implements some first-best allocation $(q^*, d^*, z_b, z_s, n_b, n_s)$ with some μ and

[9] Notice that e-money may remain essential in this region. Even though it cannot implement the first best, it may still improve the allocation.

$B_s(z_n, n_n) \leq 0$. Consider a first-best allocation (q^*, d^*, z_b', z_s') under a money mechanism $\mathcal{M} = \{B_b(z), B_s(z), \mu\}$, where $z_b' = z_b + n_b$, $z_s' = 0$, $B_s(z) = 0$ for all z, and

$$B_b(z) = \begin{cases} B_b(z_n, n_b) - A, \text{ if } z = z_b' \\ 0, \text{ otherwise,} \end{cases}$$

where

$$A \equiv \left(1 - \frac{1}{\mu}\right)(n_s + n_b + z_s + z_b) - B_s(z_n, n_n).$$

Notice that $A \geq 0$ due to the premise $B_s(z_n, n_n) \leq 0$. Then it is straightforward to verify that (15) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$ since $z_s' = B_s(z) = 0$. Also, notice that

$$\begin{aligned} & -\mu d^* - B_b(z_b) + \beta U(q^*) \\ = & -\mu d^* - B_b(z_n, n_b) + \beta U(q^*) - A \\ \geq & \max_q \{-\mu D(q) + \beta U(q)\} - A, \end{aligned}$$

where the last inequality comes from the fact that $(q^*, d^*, z_b, z_s, n_b, n_s)$ is incentive compatible to buyers under$\mathcal{M}_E$. So (13) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$. Finally, it is straightforward to verify that (16) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$. Thus (q^*, d^*, z_b', z_s') is incentive compatible to buyers and sellers under $\mathcal{M}$, and $\mathcal{M}$ is self-financed. Then it leads to contradiction as there exists a money mechanism $\mathcal{M}$ which implements the first best. Since $\mu \geq 1$, the result that $B_s(z_n, n_n) > 0$ implies $B_b(z_n, n_n) > 0$ from the e-money issuer's budget (26). ■

As mentioned above, when θ is too low, extracting trade surplus from buyers alone cannot raise enough resources to support the first best. The power of limited participation helps implement the first best by extracting surplus from sellers (i.e. $B_s(z, n) > 0$) to cross-subsidize buyers' holding of e-money balances (i.e. $B_b(z, n) < 0$). The key benefit of limiting participation is allowing cross-subsidization from sellers to buyers, which is infeasible under a money mechanism.

Simple examples

Suppose $\mu > \bar{\mu}$ and $\theta \in [\underline{\theta}, \overline{\theta})$. So according to Proposition 6, e-money with limited participation is essential. We will illustrate examples of simple direct and indirect mechanisms. In these extreme examples, sellers get zero trade surplus, but more general cases can be similarly constructed.

(i) Direct mechanism

Under this simple mechanism, the transfer function for sellers is a fixed fee:

$$B_s(z_s, n_s) = \beta[d^* - C(q^*)] \text{ for any } (z_s, n_s),$$

and the transfer function for buyers is:

$$B_b(z_s, n_b) = \begin{cases} \beta C(q^*) + (1-\beta-\mu)d^*, \text{ if } n_b = \mu d^* \\ 0, \text{ otherwise.} \end{cases}.$$

When $\mu > \bar{\mu}$, Proposition 1 implies that buyers not joining the e-money mechanism will choose not to trade. In this case, a buyer joining the e-money mechanism needs to bring $\beta C(q^*) + (1-\beta)d^*$ from the CM, receive a transfer $-\beta C(q^*) - (1-\beta-\mu)d^*$ from the issuer, and brings d^* into the DM to consume q^*. We can show that the participation constraint is satisfied when

$$\theta \geq \underline{\theta} = \frac{\overline{\theta} - \beta}{1-\beta}.$$

Notice that this scheme exhibits the features of non-linear transfers and cross-subsidization.

(ii) Indirect mechanism: fixed membership fee and proportional rewards

The e-money issuer imposes a fixed membership fee B_b on buyers who can then collect interest on their money balances at the rate R in the end of the CM:

$$\begin{aligned} R &= \frac{\mu}{\beta} - 1, \\ B_b &= \beta C(q^*) - (2\beta - 1)\, d^*. \end{aligned}$$

Without paying B_b, a buyer cannot use e-money in the next DM. Similarly, in order to receive e-money in the next DM, a seller has to pay

$$B_s = \beta[D(q^*) - C(q^*)].$$

The e-money issuer's budget is balanced. Obviously, sellers are indifferent between joining or not. Buyers have incentive to join when

$$-D(q^*)\beta - B_b + \beta U(q^*) \geq 0,$$

where $D(q^*)\beta$ is the balance they need to bring to the DM so that, after interest payment, they have real balance $D(q^*)$ to finance the efficient quantity in the DM. One can show that this is positive when $\theta \geq \overline{\theta}$. Notice that this scheme also exhibits the features of piece-wise linear transfers and cross-subsidization. This mechanism does not involve money. Appendix B considers an example involving money deposit. In that example, the e-money mechanism is designed to support positive value of money in equilibrium.

Summary

In this section, we learn that

1. An optimal e-money mechanism with limited participation is at least as good as an optimal money mechanism for any given money growth rate;
2. When buyers have moderate bargaining power and money growth rate is high, e-money mechanism with limited participation is essential, and involves cross-subsidization from sellers to buyers;
3. First-best can be implemented by a simple indirect mechanism with fixed membership fees on buyers and sellers, and proportional rewards on buyers' balances.

5 Electronic money with limited transferability

Now we consider limited transferability as an alternative feature of e-money. Suppose the e-money issuer has the power to block e-money transfers among agents in the DM. However, the e-money issuer dose not know the amount of e-money transferred nor the identities of the payer and payee. In the DM, a payer chooses to pay the e-money issuer Δ_b units of e-money in order to transfer e-money to someone else, or the e-money issuer blocks any e-money transfers. Similarly, a payee chooses to pay the e-money issuer Δ_s units of e-money in order to receive e-money from someone else, or it is blocked. We interpret Δ_b and Δ_s as interchange fees. Notice that limited transferability is different from limited participation: in a mechanism with limited participation, agents need to pay fees to join the e-money mechanism, in order to use e-money in the DM; in a mechanism with limited transferability, agents first match in the DM and then decide whether to pay the interchange fees for using e-money as the means of payment, regardless of whether they have joined the e-money mechanism before. An important distinction is that a mechanism featuring limited participation collects fees only in the CM, while a mechanism featuring limited transferability can also collect fees only in the DM. This distinction leads to different abilities to implement the first best allocation. In the CM, e-money is assumed to be freely transferable among agents.[10]

CM and DM decision

An e-money mechanism $\mathcal{M}_L$ featuring limited transferability consists of the e-money transfer functions based on the portfolio reported and fees in the DM, denoted as $\mathcal{M}_L \equiv \{\Delta_b, \Delta_s, B_b(z,n), B_s(z,n)\}$. In the DM, the buyer's value function under an e-money mechanism $\mathcal{M}_L$, $V_b(z,n)$, is

$$V_b(z,n) = U[q(z,n)] + W_b\left[z + n - d_z(z,n) - d_n(z,n) - I_{d_n(z,n)>0}\Delta_b\right], \tag{31}$$

where $\{q(z,n), d_z(z,n), d_n(z,n)\}$ is the bargaining solution as a function of the buyer's portfolio. It specifies that the buyer to pay d_z units of real money balances and d_n units of real e-money balances for q units of the DM goods produced by the seller. $I_{d_n(z,n)>0}$ is an indicator function capturing that the buyer

[10] Allowing the issuer to have the additional power to restrict transferability in the CM can only strengthen our finding.

needs to pay the fee Δ_b whenever e-money payment is positive, or $d_n(z,n) > 0$. $W_b(a)$ is the CM value function of the buyer under an e-money mechanism $\mathcal{M}_L$, which is given by

$$W_b(a) = \max_{z',n',e_b\in\{0,1\}} \left\{ a - z' - n' + e_b J_b(z',n') + (1-e_b)\beta V_b\left(\frac{z'}{\mu},\frac{n'}{\mu}\right)\right\}, \text{ s.t.} \tag{32}$$

where $J_b(z,n)$ is the buyer's value function if she chooses to participate in the mechanism, which is given by

$$J_b(z,n) = \max_{\widehat{z},\widehat{n}} \left\{ -B_b(\widehat{z},\widehat{n}) + \beta V_b\left(\frac{z}{\mu},\frac{n}{\mu}\right)\right\}$$

$$\begin{aligned} \widehat{z} &\leq z, \\ \widehat{n} &\leq n. \end{aligned}$$

In the DM, the value function, $V_s(a,e_s)$, of a seller who has joined an e-money mechanism $\mathcal{M}_L$, is

$$V_s(a,e_s) = -C[q(z_b,n_b)] + W_s\left[a + d_z(z_b,n_b) + d_z(z_b,n_b) - I_{d_n(z_b,n_b)>0}\Delta_s\right]. \tag{33}$$

Again, the seller needs to pay a fee Δ_s in order to receive the e-money payment $d_n(z_b,n_b) > 0$. In the CM, the seller's value function under an e-money mechanism $\mathcal{M}_L$, $W_s(a)$, is

$$W_s(a) = \max_{z',n',e_s\in\{0,1\}} \left\{ a - z' - n' + e_s J_s(z',n') + (1-e_s)\beta V_s\left(\frac{z'+n'}{\mu}\right)\right\}, \text{ s.t.} \tag{34}$$

$$\begin{aligned} \widehat{z} &\leq z', \\ \widehat{n} &\leq n'. \end{aligned}$$

where $J_s(z,n)$ is the CM value of the seller if she chooses to participate the mechanism, which is given by

$$J_s(z,n) = \max_{\widehat{z},\widehat{n}} \left\{ -B_s(\widehat{z},\widehat{n}) + \beta V_s\left(\frac{z+n}{\mu}\right)\right\}$$

$$\begin{aligned} \widehat{z} &\leq z, \\ \widehat{n} &\leq n. \end{aligned}$$

Bargaining solution

Given the terms-of-trade (q, d_z, d_n), the DM trade surpluses of the buyer and the seller are respectively

$$\begin{aligned} S_b(q, d_z, d_n) &= U(q) + W_b(z_b + n_b - d_z - d_n - I_{d_n>0}\Delta_b) - W_b(z_b + n_b), \\ S_s(q, d_z, d_n) &= -C(q) + W_s(a_s + d_z + d_n - I_{d_n>0}\Delta_s) - W_s(a_s). \end{aligned}$$

Under an e-money mechanism $\mathcal{M}_L$, the bargaining solution $\{q(z_b, n_b), d_z(z_b, n_b), d_z(z_b, n_b)\}$ is given by

$$\max_{q,d_z,d_n} S_b(q, d_z, d_n) + S_s(q, d_z, d_n), \text{ s.t.} \tag{35}$$

$$S_b(q, d_z, d_n) = \theta[S_b(q, d_z, d_n) + S_s(q, d_z, d_n)],$$

$$d_z \leq z_b, \tag{36}$$

$$d_n + I_{d_n>0}\Delta_b \leq n_b, \tag{37}$$

$$I_{d_n>0}\Delta_s \leq d_n. \tag{38}$$

Here, the first constraint is the pricing protocol. The second and third are respectively the buyer's liquidity constraints on money and e-money payments. The last one requires that the seller's e-money balance received is sufficient to finance the interchange fee imposed on the seller.

Using the fact that $W_b(a)$ and $W_s(a)$ are linear, the bargaining problem (35) can be reformulated as

$$\max_{q,d_z,d_n} \{U(q) - C(q) - I_{d_n>0}(\Delta_b + \Delta_s)\}, \text{ s.t. (36), (37), (38), and} \tag{39}$$

$$(1-\theta)[U(q) - d_z - d_n - I_{d_n>0}\Delta_b] = \theta[-C(q) + d_z + d_n - I_{d_n>0}\Delta_s],$$

Define $\Delta \equiv \Delta_b + \Delta_s$. It is straightforward to show the following lemma, which characterizes the bargaining solution under $\mathcal{M}_L$

The bargaining solution $\{q(z_b, n_b), d_z(z_b, n_b), d_n(z_b, n_b)\}$ satisfies

$$d_z + d_n + I_{d_n>0}\Delta_b = D(q) + I_{d_n>0}\theta\Delta,$$

$$d_z = \min\{z_b, (1-\theta)U^* + \theta C^*\}$$

$$d_n + I_{d_n>0}\Delta_b = \begin{cases} \min\{n_b, (1-\theta)U^* + \theta C^* + \theta\Delta - d_z\}, \\ \quad \text{if } n_b > \Delta_b \text{ and } U(q) - C(q) - I_{d_n>0}\Delta \geq U(q(z_b)) - C(q(z_b)) \\ 0, \text{ otherwise} \end{cases}$$

In the presence of interchange fees, there is a pecking order of payment: using money before e-money to avoid paying the interchange fee. The interchange fees partially pass through prices. If e-money is used, then from Lemma 5 the total payment made by the buyer is given by $d_z + d_n + \Delta_b = D(q) + \theta\Delta$, and the total payment received by the seller is given by $d_z + d_n - \Delta_s = D(q) + (1-\theta)\Delta$. A higher buyer's bargaining power θ will result in a higher pass through of the interchange fees Δ on the total payment made by the buyer.

Incentive-compatibility for buyers

In the CM, buyers who do not participate in the e-money mechanism $\mathcal{M}_L$ will not use e-money, since both money and e-money have the same inflation rate μ but using e-money needs to pay an extra interchange fee Δ_b. Thus, $e_b = 0$ implies $n_b = d_n = 0$. Using the linearity of $W_b(a)$, and ignoring the constant terms, one can reformulate the buyer's problem in the CM as

$$\max_{z',\widehat{z},\widehat{n},q,e_b\in\{0,1\}} \left\{ \begin{array}{c} e_b\left[-\mu\left[D(q)+\theta\Delta\right] - B_b(\widehat{z},\widehat{n}) + \beta U(q)\right] \\ +(1-e_b)\left[-\mu D(q) + \beta U(q)\right] \end{array} \right\}, \text{ s.t.} \tag{40}$$

$$\begin{aligned} \widehat{z} &\leq z' \\ \widehat{n} &\leq \mu\left[D(q)+\theta\Delta\right] - z' - \Delta_b. \end{aligned}$$

Definition 10 *An allocation $(q, d_z, d_n, z_b, z_s, n_b, n_s)$ is incentive compatible for buyers under an e-money mechanism $\mathcal{M}_L$ if $d_z = z_b/\mu$ and $d_n + \Delta_b = D(q) + \theta\Delta - d_z = n_b/\mu$, as well as $e_b = 1$, $\widehat{z} = z' = z_b$, $\widehat{n} = n_b$, and q solve (40).*

As before, to induce buyers to participate in the mechanism, it is necessary to have an incentive-compatible allocation $(q, d_z, d_n, z_b, z_s, n_b, n_s)$ satisfies

$$-B_b(z_b, n_b) - \mu\left[D(q)+\theta\Delta\right] + \beta U(q) \geq \max_{q'} \left\{-\mu D(q') + \beta U(q')\right\}. \tag{41}$$

Incentive-compatibility for buyers

Similarly, using the linearity of $W_s(a)$, and ignoring the constant terms, one can reformulate the seller's problem in the CM as

$$\max_{\substack{\widehat{z},z',\widehat{n},n',\\ e_s\in\{0,1\}}} \left\{-z' - n' + e_s\left[-B_s(\widehat{z},\widehat{n}) + \frac{\beta}{\mu}(z'+n')\right]\right\}, \text{ s.t.} \tag{42}$$

$$
\begin{aligned}
\widehat{z} &\leq z', \\
\widehat{n} &\leq n'.
\end{aligned}
$$

Definition 11 *An allocation $(q, d_z, d_n, z_b, z_s, n_b, n_s)$ is incentive compatible for sellers under an e-money mechanism $\mathcal{M}_E$ if $e_s = 1$, $\widehat{z} = z' = z_s$, $\widehat{n} = n' = n_s$ solve (42).*

To induce sellers to participate the mechanism, it is necessary to have an incentive-compatible allocation $(q, d_z, d_n, z_b, z_s, n_b, n_s)$ satisfies

$$
-\left(1 - \frac{\beta}{\mu}\right)(z_s + n_s) - B_s(z_s, n_s) \geq 0. \tag{43}
$$

E-money issuer's budget constraint

Definition 12 *An e-money mechanism $\mathcal{M}_L \equiv \{\Delta_b, \Delta_s, B_b(z,n), B_s(z,n)\}$ is self-financed with limited transferability under the allocation $(q, d_z, d_n, z_b, z_s, n_b, n_s)$ if*

$$
0 = \Delta_b + \Delta_s + B_b(z_b, n_b) + B_s(z_s, n_s) + \left(1 - \frac{1}{\mu}\right)(n_b + n_s). \tag{44}
$$

Implementability of first best

Definition 13 *An e-money mechanism $\mathcal{M}_L$ implements the first best with limited transferability if*

a. there exists $(d_z, d_n, z_b, z_s, n_b, n_s)$ such that the first-best allocation $(q^*, d_z, d_n, z_b, z_s, n_b, n_s)$ is incentive compatible to buyers and sellers; and

b. $\mathcal{M}_L$ is self-financed with limited transferability under the first-best allocation $(q^*, d_z, d_n, z_b, z_s, n_b, n_s)$.

As before, define $\Phi(\theta)$ as the solution to $\beta U(q^*) - C(q^*) = \max_q \{-\Phi D(q) + \beta U(q)\}$ given θ. The following proposition establishes when the optimal e-money mechanism featuring limited transferability is efficient.

Proposition 8 *Suppose $\mu \geq 1$, then there always exists an e-money mechanism $\mathcal{M}_L$ that implements the first best with limited transferability if and only if $\mu \geq \Phi(\theta)$.*

Proof. To prove Proposition (8), we only need to construct an e-money mechanism $\mathcal{M}_L$ that implements the first best with limited transferability. Consider a first-best allocation $(q^*, d_z, d_n, z_b, z_s, n_b, n_s)$ and an e-money mechanism $\mathcal{M}_L = \{\Delta_b, \Delta_s, B_b(z,n), B_s(z,n)\}$, where $z_s = n_s = 0$, $d_z = z_b/\mu$, $\Delta_s = d^* - C(q^*)$, $\Delta_b = U(q^*) - d^*$, and $B_s(z,n) = 0$ for all z and n. Define $\varepsilon \equiv \beta U(q^*) - C(q^*) - \max_q \{-\mu D(q) + \beta U(q)\} \geq 0$. Then fix any z_b and n_b satisfy $n_b \geq 0$, $z_b \in [0, \varepsilon/(1-\mu^{-1})]$ and $n_b + z_b = \mu(d^* + \theta\Delta)$. To satisfy (44), fix $B_b(z,n)$ as

$$
B_b(z,n) = \begin{cases} -\Delta - \left(1 - \frac{1}{\mu}\right) n_b, \text{ if } z = z_b \text{ and } n = n_b \\ 0, \text{ otherwise} \end{cases}
$$

Then it is straightforward to verify that (43) is satisfied since $B_s(z,n)=0$. To check (41), notice that

$$\begin{aligned}
& -\mu\left[D(q)+\theta\Delta\right]-B_b(z_b,n_b)+\beta U(q) \\
= & -\mu\left[D(q)+\theta\Delta\right]+\Delta+\left(1-\frac{1}{\mu}\right)\mu\left[D(q)+\theta\Delta\right]+\beta U(q)-\left(1-\frac{1}{\mu}\right)z_b \\
\geq & \ \beta U(q^*)-C(q^*)-\varepsilon \\
= & \max_{q'}\left\{-\mu D(q')+\beta U(q')\right\}.
\end{aligned}$$

Thus $(q^*, d_z, d_n, z_b, z_s, n_b, n_s)$ is incentive compatible to buyers and sellers under $\mathcal{M}_L$, and $\mathcal{M}_L$ is self-financed and implements the first best with limited transferability. ■

This proposition shows that, to implement the first best using this e-money mechanism, buyers' bargaining power and inflation need to satisfy $\mu \geq \Phi(\theta)$. It is straightforward to show that $\Phi(\theta)$ is increasing in θ. The idea is that an increase in θ raises the value of buyers' outside option of non-participation. A higher inflation is needed to induce them to join the mechanism. Therefore, e-money featuring limited transferability can implement the first best when inflation is not too low.

Essentiality of limited transferability

We first show that an optimal mechanism with limited transferability is at least as good as one with limited participation.

Proposition 9 *If there exists an e-money mechanism $\mathcal{M}_E$ that implements the first best with limited participation, then there also exists an e-money mechanism $\mathcal{M}_L$ that implements the first best with limited transferability under the same μ.*

Proof. Since $\mathcal{M}_E$ implements the first best, from the proof of Proposition 4 it is necessary to have

$$\begin{aligned}
\max_q\left\{-\mu D(q)+\beta U(q)\right\} & \leq \beta\left[d^*-C(q^*)\right]-d^*+\beta U(q^*) \\
& = -(1-\beta)\left[d^*-C(q^*)\right]+\beta U(q^*)-C(q^*) \\
& \leq \beta U(q^*)-C(q^*).
\end{aligned}$$

Thus we have $\theta \geq \Theta(\theta,\mu)$, then from Proposition 4 there exists an e-money mechanism $\mathcal{M}_E$ that implements the first best. ■

Proposition 5 then implies that, fixing the money growth rate, an optimal e-money mechanism featuring limited transferability is at least as good as an optimal money mechanism in implementing the first best allocation.

Proposition 10 (Essentiality of e-money with limited transferability) *If $\theta < \underline{\theta}$, then first best allocation can be implemented by an e-money mechanism with limited transferability when $\mu \geq \bar{\mu}$.*

Proof. Omitted here. ∎

This proposition establishes the essentiality of e-money featuring limited participation. By Proposition 2 and 6, first best cannot be supported by any money mechanism or e-money mechanism featuring limited participation. However, an e-money mechanism with limited transferability can still achieve the first best when the inflation is sufficiently high. Why limited transferability is more powerful than limited participation? Recall that limited participation allows the issuer to use exclusion from period $t+1$ DM as a threat to enforce fees in period t. That is why the maximum surplus extractable from a seller is $\beta[d^* - C(q^*)]$, which is discounted because the fee is paid a period in advance. In contrast, the ability to limit transferability allows an issuer to extract seller's trade surplus in period t DM by enforcing interchange fees in the same period. As a result, the maximum surplus extractable from a seller becomes $d^* - C(q^*)$, without discounting. Therefore, postponing the fee collection helps relax the seller's participation constraint. But will it tighten the issuer's budget constraint? No, because the issuer can always create more e-money balances when needed. From the issuer's point of view, collecting the fee in the CM or in the following DM does not matter, as long as the money growth rate between two CM markets can be maintained at μ. In particular, the issuer can temporarily create extra e-money balances in CM, and undo it later when interchange fees are collected in the following DM. Therefore, limited transferability allows the e-money issuer to postpone fee collection, maximizing surplus extraction, without tightening its budget constraint.[11]

Characterization of optimal e-money mechanism

After establishing the essentiality of e-money, we now characterize the optimal e-money mechanism.

Proposition 11 *Suppose $\theta < \Theta(\theta, \mu)$ (i.e., first best not implementable by any e-money mechanism with limited participation). If there exists an e-money mechanism $\mathcal{M}_L = \{\Delta_b, \Delta_s, B_b(z,n), B_s(z,n)\}$ that implements the first best with limited transferability, then $\Delta > 0$ and $B_b(z_n, n_n) < 0$.*

Proof. Suppose not, i.e., there exists an e-money mechanism $\mathcal{M}_L = \{\Delta_b, \Delta_s, B_b(z,n), B_s(z,n)\}$ that implements some first-best allocation $(q^*, d_z, d_n, z_b, z_s, n_b, n_s)$ with limited transferability under $\Delta = 0$ and some μ, but there does not exists any e-money mechanism $\mathcal{M}_E = \{B_b(z,n), B_s(z,n)\}$ that implements the first-best allocation with limited participation under the same μ. Given $\Delta = 0$, consider a money mechanism$\mathcal{M}_L = \{B_b'(z), B_s'(z), \mu\}$ where

$$B_s'(z) = \begin{cases} B_s(z_s, n_s), \text{ if } z = z_s + n_s \\ 0, \text{ otherwise} \end{cases}$$

[11] Note that $\partial\underline{\theta}/\partial\beta > 0$, implying that limited transferability is more essential relative to limited participation when discount factor is low. A real word interpretation is that charging interchange fees at the time of transaction is more desirable relative to charging a membership fee in advance when the frequency of membership fee payment is low (e.g. annual membership paid a year in advance).

$$B_b'(z) = B_b(z,n) - \left(1-\mu^{-1}\right)(z_b+z_s)$$

Consider a first-best allocation (q^*, d^*, z_b', z_s') where $z_b' = z_b + n_b$ and $z_s' = z_s + n_s$. Then it is straightforward to verify that (15) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$ since (15) and (43) are the same. Also, notice that

$$\begin{aligned} & -\mu d^* - B_b(z_b) + \beta U(q^*) \\ = & -\mu d^* - B_b(z_b, n_b) + \beta U(q^*) + \left(1-\mu^{-1}\right)(z_b+z_s) \\ \geq & \max_q \{-\mu D(q) + \beta U(q)\}, \end{aligned}$$

where the last inequality comes from the fact that $(q^*, d^*, z_b, z_s, n_b, n_s)$ is incentive compatible to buyers under$\mathcal{M}_L$. So (13) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$. Finally, it is straightforward to verify that (16) is satisfied under the first-best allocation (q^*, d^*, z_b', z_s') with $\mathcal{M}$. Thus (q^*, d^*, z_b', z_s') is incentive compatible to buyers and sellers under $\mathcal{M}$, and $\mathcal{M}$ is self-financed. Then it leads to contradiction to Proposition 5 as there exists a money mechanism $\mathcal{M}$ which implements the first best but there does not exist any e-money mechanism $\mathcal{M}_E = \{B_b(z,n), B_s(z,n)\}$ that implements the first-best allocation with limited participation under the same μ. Thus we establish $\Delta > 0$. Finally, notice that (9) is satisfied only if $B_b(z_n, n_n) < 0$. Thus we prove Proposition 11. ■

As discussed above, to implement the first best, the issuer has to extract trade surplus in the DM ($\Delta > 0$), which is then used to induce buyers to carry sufficient e-money balances in the CM ($B_b(z_n, n_n) < 0$). Note that, this scheme requires the issuer to temporarily expand e-money supply in the CM (to pay buyers $-B_b(z_n, n_n)$) and later undo it in the DM (by charging fees Δ), ensuring a constant money growth across periods.

Simple examples

Suppose $\mu > \bar{\mu}$. We will illustrate examples of simple direct and indirect mechanisms. In these extreme examples, sellers get zero trade surplus, but more general cases can be similarly constructed.

(i) Direct mechanism

Under this simple mechanism, the transfer functions are

$$\begin{aligned} B_b(z_s, n_b) &= \begin{cases} C(q^*) - \mu d_n^*, \text{ if } n_b = \mu d_n^* \\ 0, \text{ otherwise.} \end{cases} \\ B_s(z_s, n_s) &= 0, \text{ for any } (z_s, n_s), \end{aligned}$$

where $d_n^* = D(q^*) + \theta\Delta_s$, and the interchange fees are

$$\begin{aligned} \Delta_b &= 0, \\ \Delta_s &= d^* - C(q^*) \end{aligned}$$

The budget constraint of the issuer is satisfied. Obviously, the seller's participation constraint is satisfied. When $\mu > \bar{\mu}$, Proposition 1 implies that buyers not joining the e-money mechanism will choose not to trade. In this case, a buyer has an incentive to join the e-money mechanism to bring d_n^* into the DM to consume q^* if $-C(q^*) + \beta U(q^*) \geq 0$, which is always satisfied. This scheme features cross-subsidization from sellers to buyers, with non-linear pre-trade transfers to buyers, and post-trade fees on sellers.

(ii) Indirect mechanism: fixed membership fee, proportional rewards, and interchange fee on merchants

The e-money issuer imposes a fixed membership fee B_b on buyers who can then collect interest on their money balances at the rate R in the end of the CM:

$$\begin{aligned} R &= \frac{\mu}{\beta} - 1, \\ B_b &= C(q^*) - \beta d_n^*, \end{aligned}$$

where $d_n^* = D(q^*) + \theta\Delta_s$. Also, the seller has to pay an interchange fee

$$\Delta_s = d_n^* - C(q^*)$$

Obviously, sellers are indifferent between joining or not. The issuer's budget is balanced. Buyers have incentive to join when

$$-\beta d_n^* - B_b + \beta U(q^*) \geq 0,$$

where βd_n^* is the balance they need to acquire in the CM so that, after interest payment, they have real balance d_n^* to finance the efficient quantity in the DM. One can show that this is positive when $-C(q^*) + \beta U(q^*) \geq 0$. This scheme features cross-subsidization from sellers to buyers, with piece–wise linear pre-trade transfers to buyers, and post-trade fees on sellers. This mechanism does not involve money. Appendix C gives an example involving money deposit. In that example, the e-money mechanism is designed to support positive value of money in equilibrium.

Summary

In this section, we learn that

1. An optimal e-money mechanism with limited transferability is at least as good as any money mechanism and e-money mechanism with limited participation for any given money growth rate;

2. When buyers have low bargaining power and money growth rate is high, e-money mechanism with limited transferability is essential, and involves cross-subsidization from sellers to buyers using interchange fees;

3. First-best can be implemented by a simple indirect mechanism with fixed membership fees on buyers, proportional rewards on buyers' balances, and interchange fees on sellers.

6 Concluding Remarks

Returning to our initial question: what's special about e-money? Our model predicts that an optimally designed e-money system should carry such pricing features as membership fees, interchange fees, and rewards to buyers. This prediction does have some empirical support in the sense that some successful real-world e-money systems (e.g. Octopus cards in Hong Kong) also exhibit these features. But let us point out that these pricing schemes are not really revolutionary breakthroughs that electronic money bring about. After all, many conventional, account-based payments systems (e.g. retail banking) have long adopted these features. Most of these systems, however, involve centralized arrangement of accounts with identified owners, making it inaccessible to some population and infeasible in some situation. In contrast, the major breakthrough that electronic money brings about is its ability to support these features by limiting participation and transferability even in a decentralized setting with anonymous users – a setting renders cash essential. We argue that it is this breakthrough that makes e-money special, supporting welfare-improving non-linear pricing schemes and cross-subsidization, mitigating fundamental frictions (i.e. liquidity constraint and holdup problems) that often impair the efficiency of a money-based system.

Appendix

A Implementation by Indirect Mechanisms: Interest-Bearing Money

So far we have exploited the power of revelation principle and focus on the set of direct mechanisms which implements the first-best allocation when $\theta \geq \overline{\theta}$. In general the reverse of revelation principle is not true: it is possible to have some first-best allocations which can be implemented by a direct mechanism, such as the one constructed above, but not by indirect mechanisms. However, we will show in this section how an indirect mechanism based on the one proposed by Andolfatto (2009) can be used to implement the first best when $\theta \geq \overline{\theta}$.

Like the mechanism suggested by Andolfatto (2009), consider now the money issuer charges buyers a fixed membership fee B to collect interest on money at the rate R in the end of the CM. The mechanism has nothing to do with sellers. Thus, an Andolfatto's mechanism is indexed by a triple $\mathcal{M}_A \equiv \{B, R, \mu\}$. The optimization problem of a buyer in the CM under an Andolfatto's mechanism can be formulated as

$$\max_{z', q, e \in \{0,1\}} \{e\left[-z' - B + \beta U(q)\right] + (1-e)\left[-\mu D(q) + \beta U(q)\right]\}, \text{ s.t.} \tag{A.1}$$

$$D(q) = \frac{1+R}{\mu} z'. \tag{A.2}$$

In the equilibrium, we have $z_b = z'$ and $z_s = 0$.

Definition 14 *An Andolfatto's mechanism* $\mathcal{M}_A \equiv \{B, R, \mu\}$ *is self-financed under the allocation* $(q, d, z_b, 0)$ *if*

$$Rz_b = B + (\mu - 1) z_b. \tag{A.3}$$

Definition 15 *An Andolfatto's mechanism* $\mathcal{M}_A$ *implements the first best if*

a. $z' = \frac{\mu d^*}{1+R} = z_b$, $q = q^*$ *and* $e = 1$ *solves (A.1); and*

b. $\mathcal{M}_A$ *is self-financed under the first-best allocation* $(q^*, d^*, z_b, 0)$*.*

Define μ_0 as solution $\mu = \mu_0$ solving

$$-d^* + \beta U(q^*) = \max_{q'} \{-\mu D(q') + \beta U(q')\}. \tag{A.4}$$

The following lemma shows when μ_0 is well-defined

Lemma 2 *There exists* μ_0 *solving (A.4) if and only if* $\theta \geq \overline{\theta}$*. Also,* $\mu_0 > 1$ *if exists.*

Proof. Omitted here. ■

The following proposition characterizes the set of Andolfatto's mechanism which implements the first best.

Proposition 12 *Suppose $\theta \geq \overline{\theta}$. There exists an Andolfatto's mechanism which implements the first best, which is constructed as follows*

a. $\mu \geq \mu_0$*;*

b. $R = \frac{\mu}{\beta} - 1$*; and*

c. $B = \mu (1 - \beta) d^*$*.*

Proof. Omitted. ■

B Implementation by Indirect Mechanisms with limited participation: Membership-Reward-Deposit E-money

Proposition 4 states that when $\theta \geq \Theta(\theta, \mu)$, there exists a non-empty set of direct mechanisms which implements the first-best allocation with limited participation. We are also interested to construct some simple indirect mechanisms which can implement the first-best allocation with limited participation. Consider now the e-money issuer charges sellers a fixed membership fee B_s to use e-money in the coming DM. To use e-money, buyers have to maintain a deposit of at least z_b units of real money balances, for a return in terms of a fixed reward $-B_b$ units of real e-money balances, and a proportional reward at a rate R to load e-money in the CM. The deposit can be used in the DM. Thus, a membership-reward-deposit mechanism is indexed by $\mathcal{M}_M \equiv \{B_s, B_b, R, z_b\}$. The optimization problem of a buyer in the CM under a membership-reward-deposit mechanism can be formulated as

$$\max_{z',n',q,q',e_b \in \{0,1\}} \{e[-z' - n' + \beta U(q)] + (1-e)[-\mu D(q') + \beta U(q')]\}, \text{ s.t.} \tag{B.1}$$

$$\begin{aligned} D(q) &= \frac{z' - B_b}{\mu} + \frac{1+R}{\mu} n', \\ z' &\geq z_b. \end{aligned}$$

The optimization problem of a seller in the CM under a membership-reward-deposit mechanism can be formulated as

$$\max_{e_s \in \{0,1\}} \left\{ e[-B_s + \beta[d - C(q)]] + (1-e)\beta\left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right]\right\}, \text{ s.t.} \tag{B.2}$$

Definition 16 *A membership-reward-deposit mechanism $\mathcal{M}_M \equiv \{B_s, B_b, R, z_b\}$ is self-financed under the allocation $(q, d, z_b, 0, n_b, 0)$ if*

$$0 = B_s + B_b - \frac{R}{1+R}(n_b + B_b) + \left(1 - \mu^{-1}\right) n_b. \tag{B.3}$$

Definition 17 *A membership-reward-deposit mechanism $\mathcal{M}_M$ implements the first best if*

a. $z' = z_b$, $n' = \frac{n_b + B_b}{1+R}$, $q = q^*$ and $e_b = 1$ solves (B.1);

b. $e_s = 1$ solves (B.2);

c. $\mathcal{M}_M$ is self-financed under the first-best allocation $(q^*, d^*, z_b, 0, n_b, 0)$.

Define $\varepsilon_0 \equiv -(1-\beta)d^* + \beta[U(q^*) - C(q^*)] - \max_q\{-\mu D(q) + \beta U(q)\}$, where $\varepsilon_0 \geq 0$ if and only if $\theta \geq \Theta(\theta, \mu)$. The following proposition characterizes the set of membership-reward-deposit mechanisms which implements the first best.

Proposition 13 *Suppose $\theta \geq \Theta(\theta, \mu)$ and $\mu \geq 1$. There exists a membership-reward-deposit mechanism which implements the first best, which is constructed as follows*

a. $R = \frac{\mu}{\beta} - 1$;

b. *any* $n_b > 0$ *and* $z_b > 0$ *such that* $n_b + z_b = \mu d^*$ *and* $\beta\left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] + \left(1 - \frac{1}{\mu}\right) z_b \leq \varepsilon_0$;

c. $B_s \in \left[0, \beta\left[d^* - C(q^*)\right] + \left(1 - \frac{1}{\mu}\right) z_b - \varepsilon_0\right]$; *and*

d. $B_b = -\mu B_s/\beta - \left(1 - \beta^{-1}\right) n_b$.

Proof. First, notice that

$$
\begin{aligned}
& \beta\left[d^* - C(q^*)\right] + \left(1 - \frac{1}{\mu}\right) z_b - \varepsilon_0 \\
\geq\ & \beta\left[d^* - C(q^*)\right] + \left(1 - \frac{1}{\mu}\right) z_b - \beta\left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] - \left(1 - \frac{1}{\mu}\right) z_b \\
=\ & \beta\left[d^* - C(q^*) - d\left(\frac{z_b}{\mu}\right) + C\left[q\left(\frac{z_b}{\mu}\right)\right]\right] \geq 0,
\end{aligned}
$$

so the set $\left[0, \beta\left[d^* - C(q^*)\right] + \left(1 - \frac{1}{\mu}\right) z_b - \varepsilon_0\right]$ in (c) is well-defined. Combining (b) and (c), we have

$$
-B_s + \beta\left[d^* - C(q^*)\right] \geq \beta\left[d\left(\frac{z_b}{\mu}\right) - C\left[q\left(\frac{z_b}{\mu}\right)\right]\right],
$$

so $e_s = 1$ satisfies (B.2). Notice that under the membership-reward-deposit mechanism. Also, substituting $D(q) = \frac{z' - B_b}{\mu} + \frac{1+R}{\mu} n'$ into (B.1), we have (B.1) equivalent to

$$
\max_{\substack{z' \geq z_b, q, \\ q', e_b \in \{0,1\}}} \left\{ e\left[-\left(1 - \frac{\beta}{\mu}\right) z' - \frac{\beta}{\mu} B_b + \beta\left[U(q) - D(q)\right]\right] + (1 - e)\left[-\mu D(q') + \beta U(q')\right]\right\}, \text{ s.t.} \tag{B.4}
$$

$n' = \frac{n_b + B_b}{1+R}$, $z' = z_b$ and $q = q^*$ solve the above. Substituting (c) and (d) into $-\left(1 - \frac{\beta}{\mu}\right) z_b - \frac{\beta}{\mu} B_b + \beta\left[U(q^*) - D(q^*)\right]$, we have

$$
\begin{aligned}
& -\left(1 - \frac{\beta}{\mu}\right) z_b - \frac{\beta}{\mu} B_b + \beta\left[U(q^*) - D(q^*)\right] \\
\geq\ & -\left(\frac{1 - \beta}{\mu}\right)(z_b + n_b) + \beta\left[U(q^*) - C(q^*)\right] - \varepsilon_0 \\
=\ & \max_q \left\{-\mu D(q) + \beta U(q)\right\}.
\end{aligned}
$$

So $e_b = 1$ satisfies (B.1). Finally, the construction of B_b from (d) implies (B.3) is satisfied. Thus, any membership-reward-deposit mechanism satisfies (a) to (d) implements the first best. ∎

C Implementation by Indirect Mechanisms with Limited Transferability: Interchange-Reward-Deposit E-money

Proposition 8 states that when $\mu \geq \Phi(\theta)$, there exists non-empty set of direct mechanisms which implements the first-best allocation with limited transferability. We are also interested to construct some simple indirect mechanisms which can implement the first-best allocation with limited transferability. Consider now to use e-money, buyers have to maintain a deposit of at least z units of real money balances, for a return in terms of a fixed reward $-B$ units of real e-money balances, and a proportional reward at a rate R to load e-money in the CM. The deposit can be used in the DM. To receive any positive amount of e-money in the DM, the payee is charged a fixed interchange fee of Δ units of real e-money balances from the e-money received. Thus, an interchange-reward-deposit mechanism is indexed by $\mathcal{M}_I \equiv \{\Delta, B, R, z\}$. The optimization problem of a buyer in the CM under an interchange-reward-deposit mechanism can be formulated as

$$\max_{z',n',q,q',e_b\in\{0,1\}} \{e[-z'-n'+\beta U(q)] + (1-e)[-\mu D(q') + \beta U(q')]\}, \text{ s.t.} \tag{C.1}$$

$$\begin{aligned} D(q) + \theta\Delta &= \frac{z'-B}{\mu} + \frac{1+R}{\mu} n', \\ z' &\geq z. \end{aligned}$$

Definition 18 *A interchange-reward-deposit mechanism $\mathcal{M}_I \equiv \{\Delta, B, R, z\}$ is self-financed under the allocation (q, d, z, n) if*

$$0 = \Delta + B - \frac{R}{1+R}(n+B) + \left(1-\mu^{-1}\right) z. \tag{C.2}$$

Definition 19 *A interchange-reward-deposit mechanism $\mathcal{M}_I$ implements the first best if*

a. $z' = z$, $n' = \frac{n+B_b}{1+R}$, $q = q^*$ and $e_b = 1$ solves (C.1);

b. $\mathcal{M}_I$ is self-financed under the first-best allocation (q^*, d^*, z, n).

Define $\varepsilon_0 \equiv \beta U(q^*) - C(q^*) - \max_q \{-\mu D(q) + \beta U(q)\}$, where $\varepsilon_0 \geq 0$ if and only if $\mu \geq \Phi(\theta)$. The following proposition characterizes the set of interchange-reward-deposit mechanisms which implements the first best.

Proposition 14 *Suppose $\mu \geq \Phi(\theta)$. There exists a interchange-reward-deposit mechanism which implements the first best, which is constructed as follows*

a. $R = \frac{\mu}{\beta} - 1$;

b. any $\Delta \in [0, U(q^*) - C(q^*)]$, $n > 0$ *and* $z > 0$ *such that* $n + z = \mu(d^* + \theta\Delta)$ *and*

$$\left(1 - \frac{1}{\mu}\right) z + (1-\theta)\left[U(q^*) - C(q^*) - \Delta\right] \leq \varepsilon_0;$$

c. $B = -(1+R)\Delta - \left(1 - \beta^{-1}\right) n.$

Proof. Substituting $D(q) = \frac{z' - B}{\mu} + \frac{1+R}{\mu} n'$ into (C.1), we have (C.1) equivalent to

$$\max_{\substack{z' \geq z, q, \\ q', e_b \in \{0,1\}}} \left\{ e\left[-\left(1 - \frac{\beta}{\mu}\right) z' - \frac{\beta}{\mu} B + \beta\left[U(q) - D(q)\right]\right] + (1-e)\left[-\mu D(q') + \beta U(q')\right] \right\}.$$

Then $n' = \frac{n+B}{1+R}$, $z' = z$ and $q = q^*$ solve the above. Together with (c), we have

$$\begin{aligned}
& -z' - n' + \beta U(q^*) \\
= & -z - \frac{n+B}{1+R} + \beta U(q^*) \\
= & -\left(1 - \frac{1}{\mu}\right) z - (1-\theta)\left[U(q) - C(q^*) - \Delta\right] + \beta U(q) - C(q^*) \\
\geq & -\varepsilon_0 + \beta U(q) - C(q^*) \\
= & \max_q \left\{-\mu D(q) + \beta U(q)\right\}
\end{aligned}$$

So $e_b = 1$ satisfies (C.1). Finally, the construction of B from (c) implies (C.2) is satisfied. Thus, any intertechange-reward-deposit mechanism satisfying (a) to (c) implements the first best. ■

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